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Reservoir Computing with Stability Transformation Method to Detect Unstable Fixed Point of Chaotic Map

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Abstract—Detecting unstable periodic orbits in chaotic systems based on the time series is a fundamental problem in nonlinear dynamics, but it often becomes extremely challenging one. In this study, we propose a new approach for detecting unstable fixed point using reservoir computing and stability transformation method. We connects reservoir computing, a well-known machine learning technique, and stability transformation method, which can detects unstable periodic orbits in chaotic dynamical systems, to perform unstable fixed point detection in a data-driven and model-free process. In this paper, we use an example of an Hénon map to demonstrate detecting unstable fixed point and unstable 2-periodic points.

1. Introduction

The study of nonlinear dynamical systems is very important not only as a fundamental problem but also from an engineering point of view. Unstable oscillatory phenomena that occur in nonlinear dynamics are called chaos, and in particular, nonlinear dynamical systems that exhibit chaos are called chaotic dynamical system. The Unstable Periodic Orbits(abbr. UPOs) inherent in chaotic attractors are an important dynamical property of chaotic dynamical systems, and many features of the dynamics can be calculated, such as Lyapunov exponents, fractal dimension, and attractor entropy [1, 2]. Because chaotic behavior exists in a wide variety of dynamical systems, periodic orbit theory has many applications and is relevant to many different fields. However, finding UPOs from chaotic attractors is difficult, both numerically and experimentally, even for simple dynamical systems, due to their instability. In particular, when the dynamics of the dynamical system and the location of the UPOs are unknown, the detection of UPOs becomes even more difficult.

In contrast, a stability transformation method (abbr. STM) [3, 4] called has been proposed to search for UPOs whose positions are unknown. The basic idea of this method is to transform a discrete chaotic dynamical system into a new dynamical system in which the UPOs are changed from unstable to stable without changing their spatial positions. Periodic orbits can be detected by iterating the transformed dynamical system.

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In our research, we proposed a chaos control method based on STM, and demonstrated its effectiveness by applying it to a simple chaos generator circuit [5]. This method is relatively simple and can stabilize UPOs with unknown position information without constraints such as the odd number limitation [6] in delayed feedback control. Furthermore, We have also proposed an improved method for robustly stabilizing high-periodic orbits [7], and an improved method for stabilizing UPOs in high-dimensional and continuous-time systems [8]. However, this control method and stability transformation method require information on the dynamics of the chaotic dynamical system, nd cannot detect and control UPOs using only time series information of the dynamical system.

In this paper, we propose a Unstable Fixed Point (abbr. UFP) detection method using Reservoir Computing (abbr. RC) and stability transformation method. The proposed method uses RC to recover a chaotic dynamical system from time series information, and applies a chaos control method based on STM to the recovered system to detect UFP in the recovered system. This method does not require information on the location of the UFP or the state of the dynamical system. The method can also be applied to a composite map to detect UPOs.

In Sect. 2, system identification by RC is described, and the proposed system is constructed by combining RC and STM. In Sect. 3, we use the Hénon map as an example to detect UFP and unstable 2-period points. Finally, conclusions and future work are given in Sect. 4.

2. Proposed method: RC and STM are used to construct the proposed system

2.1. Restoration of dynamical systems using Reservoir Computing

First, consider the following discrete chaotic system,

$$x(n+1) = Fx(n), \tag{1}$$

where n is the discrete time, $x(n)$ is the state variable of the system, and F is the discrete-time chaos map. Our goal is to search for UPOs embedded in the chaotic attractor of this system without knowing the specific state of the system (1) or the location of the UPOs.

In this paper, we use the following equation, RC[10], which is expressed as follows,

$$\boldsymbol{r}(n+1) = (1-\rho)\,\boldsymbol{r}(n) + \rho\tanh\left(\boldsymbol{W}^{in}u(n) + \boldsymbol{W}^{r}\boldsymbol{r}(n)\right), \quad (2)$$

$$o(n+1) = \boldsymbol{W}^{out}\boldsymbol{r}(n+1), \quad (3)$$

where $\boldsymbol{r}(n) = [r_1(n), \cdots, r_N(n)]^T \in \mathbb{R}^{N\times 1}$ is the reservoir state formed by N neurons at time n, $u(n) \in \mathbb{R}$ is the input signal and $o(n) \in \mathbb{R}$ is the output. $\boldsymbol{W}^{in} \in \mathbb{R}^{N\times 1}$ is the input weight matrix, $\boldsymbol{W}^{r} \in \mathbb{R}^{N\times N}$ is the joint weights among neurons in the reservoir layer, $\boldsymbol{W}^{out} \in \mathbb{R}^{1\times N}$ is the output weight matrix. The $\tanh(-)$ is an element-wise nonlinear function called the activation function. $\boldsymbol{W}^{in} \in \mathbb{R}^{N\times 1}$ and $\boldsymbol{W}^{r} \in \mathbb{R}^{N\times N}$ are determined randomly and $\boldsymbol{W}^{out} \in \mathbb{R}^{1\times N}$ are determined by learning with a teacher signal. The leak rate $\rho \in (0, 1]$ is the rate at which the reservoir forgets past information. A block diagram of this RC is shown in Fig. 1.

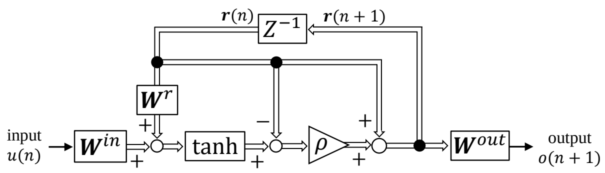

Figure 1: Block diagram of RC used in this paper.

This section describes system restoration using this RC. The basis of system restoration is 1-step prediction, and the goal is to learn the correspondence between $\boldsymbol{x}(n)$ and $\boldsymbol{x}(n+1)$ in (1). In other words, $\boldsymbol{W}^{out}$ is learned using a teacher signal, and if the output is $o(n+1) \simeq x(n+1)$ when the RC input is $u(n) = x(n)$, the system has been restored. The training data used for system restoration should be time series data that contain sufficient attractor information and omit transient states. When the number of steps in the training data is T, the input signal sequence $\boldsymbol{U} = [u(1), u(2), \cdots, u(T)]$, the driven reservoir state sequence $\boldsymbol{R} = [\boldsymbol{r}(1), \boldsymbol{r}(2), \cdots, \boldsymbol{r}(T)]$, the teacher signal sequence $\boldsymbol{Y}^{teach} = \left[y^{teach}(1), y^{teach}(2), \cdots, y^{teach}(T)\right]$, the output weights are learned by linear regression using as

$$\boldsymbol{W}^{out} = \boldsymbol{Y}^{teacher}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{R}^T)^{-1}. \quad (4)$$

The system restored by the above RC can be described by

$$\begin{aligned}\boldsymbol{r}(n+1) &= F_r(\boldsymbol{r}(n)) \\ &= (1-\rho)\,\boldsymbol{r}(n) + \rho\tanh\left(\boldsymbol{W}^{in}\boldsymbol{W}^{out}\boldsymbol{r}(n) + \boldsymbol{W}^{r}\boldsymbol{r}(n)\right).\end{aligned} \quad (5)$$

The output of (5) is $o(n) = \boldsymbol{W}^{out}\boldsymbol{r}(n)$. The block diagram of the restored system (5) is shown in Fig. 2.

2.2. Conbination of RC and STM

2.2.1. Detect for UFP

Next, we describe a proposed method that combines the above RC and STM to detect a UFP for the identified system. In this method, by applying the chaos control methods [9] to (5), an Reservoir Control System with a stable fixed point at the same position as the UFP of (5) is constructed, and UFP are searched by driving the system. The dynamics of Reservoir Control System can be described as follows,

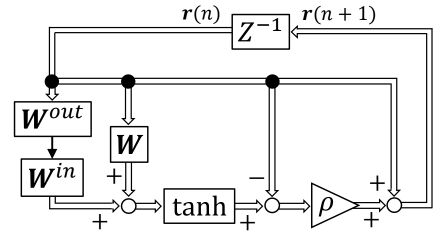

Figure 2: Block diagram of a system restored by RC.

$$\begin{cases}\boldsymbol{r}(n+1) = F_r(\boldsymbol{v}(n+1)), \\ \boldsymbol{v}(n+1) = \boldsymbol{r}(n) - \boldsymbol{K}\left(\boldsymbol{r}(n) - \boldsymbol{v}(n)\right),\end{cases} \quad (6)$$

where $\boldsymbol{v}(n) = [v_1(n), \cdots, v_N(n)]^T \in \mathbb{R}^{N\times 1}$ is the internal state of the system at time n and $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ is the control gain matrix. The output of Reservoir Control System (6) is $o(n) = \boldsymbol{W}^{out}\boldsymbol{r}(n)$. Figure 3 illustrates the block diagram of (6). The Z^{-1} in the Fig. 3 is a delay unit.

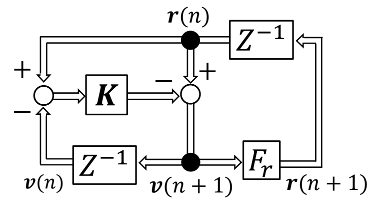

Figure 3: Block diagram of Reservoir Control System for UFP detection.

Then, the control gain $\boldsymbol{K}$ for stabilizing the UFP is derived. From the relation $\boldsymbol{r}(n) = F_r(\boldsymbol{v}(n))$, we can reduce the Eq. (6) to the following 1-D dynamics:

$$\boldsymbol{v}(n+1) = (\boldsymbol{I} - \boldsymbol{K})F_r(\boldsymbol{v}(n)) + \boldsymbol{K}\boldsymbol{v}(n). \quad (7)$$

The local linearized system of (7) in the reservoir state at fixed point $\boldsymbol{r}_{fp}$ is given by

$$\delta\boldsymbol{v}(n+1) = \boldsymbol{A}\delta\boldsymbol{v}(n), \quad (8)$$

where $\delta\boldsymbol{v}(n) \equiv \boldsymbol{v}(n) - \boldsymbol{r}_{fp}$ represents the small displacement around $\boldsymbol{r}_{fp}$. The square matrix $\boldsymbol{A}$ is the slope of F_r around $\boldsymbol{r}_{fp}$ and is written as following,

$$\boldsymbol{A} = \left.\frac{\partial\boldsymbol{v}(n+1)}{\partial\boldsymbol{v}(n)}\right|_{\boldsymbol{v}(n)=\boldsymbol{r}_{fp}} = (\boldsymbol{I} - \boldsymbol{K})DF_r(\boldsymbol{r}_{fp}) + \boldsymbol{K}, \quad (9)$$

where, $\boldsymbol{I}$ is a unitary matrix and $DF_r(\boldsymbol{r}_{fp})$ in the formula is the Jacobi matrix in $\boldsymbol{r}_{fp}$ of F_r, namely,

$$DF_r(\boldsymbol{r}_{fp}) = \left.\frac{\partial F_r(\boldsymbol{r}(n))}{\partial\boldsymbol{r}(n)}\right|_{\boldsymbol{r}(n)=\boldsymbol{r}_{fp}} = \left.\frac{\partial\boldsymbol{r}(n+1)}{\partial\boldsymbol{r}(n)}\right|_{\boldsymbol{r}(n)=\boldsymbol{r}_{fp}}. \quad (10)$$

The fixed point of (7) is stable if the absolute value of all eigenvalues of $\boldsymbol{A}$ is less than 1. In particular, if the control gain $\boldsymbol{K}$ is set to

$$\boldsymbol{K} = -(\boldsymbol{I} - DF_r(\boldsymbol{r}_{fp}))^{-1} DF_r(\boldsymbol{r}_{fp}), \tag{11}$$

the fixed point of (7) becomes super-stable because all eigenvalues of $\boldsymbol{A}$ are zero.

2.2.2. Detect for UPOs

We will now explain how to detect unstable l-periodic points using the proposed method. A periodic point p_i can be expressed as $p_i = F^l(p_i)$. Since F^l is an l-fold composite map of F and a fixed point p_f of F can be expressed as $p_f = F(p_f)$, p_i corresponds to a fixed point of F^l. Therefore, stabilizing the fixed point of the l-composite map of the restored system can also be applied to stabilize l-periodic points. The detection of UPOs in the restoration system is performed by the Reservoir Control System in Fig. 4. The dynamics of this system is described by

$$\begin{cases} \boldsymbol{r}(n+1) = F_r(\boldsymbol{v}(n+1)), \\ \boldsymbol{v}(n+1) = \boldsymbol{r}(n) - \boldsymbol{K}\left(\boldsymbol{r}(n) - \boldsymbol{v}(n)\right), \end{cases} \quad \text{for} \quad n = kl,$$
$$\begin{cases} \boldsymbol{r}(n+1) = F_r(\boldsymbol{r}(n)), \\ \boldsymbol{v}(n+1) = \boldsymbol{v}(n), \end{cases} \quad \text{for} \quad n \neq kl. \tag{12}$$

The output of this system (13) is $o(n) = \boldsymbol{W}^{out}\boldsymbol{r}(n)$. Here, focusing on the state $\boldsymbol{r}(kl), \boldsymbol{v}(kl)$ at $n = kl$, a two-dimensional map from $(\boldsymbol{r}(kl), \boldsymbol{v}(kl))$ to $(\boldsymbol{r}((k+1)l), \boldsymbol{v}((k+1)l))$ is given by

$$\begin{cases} \boldsymbol{r}((k+1)l) = F_r^l(\boldsymbol{v}((k+1)l)), \\ \boldsymbol{v}((k+1)l) = \boldsymbol{r}(kl) - \boldsymbol{K}\left(\boldsymbol{r}(kl) - \boldsymbol{v}(kl)\right), \end{cases} \tag{13}$$

where k is any positive integer. Similarly to (8), the following 1-D return map can be derived.

$$\boldsymbol{v}((k+1)l) = (\boldsymbol{I} - \boldsymbol{K})F_r^l(\boldsymbol{v}(kl)) + \boldsymbol{K}\boldsymbol{v}(kl). \tag{14}$$

In the same way as in Sect. 2, local linearization of (14) leads to the following:

$$\boldsymbol{A}_l = \left.\frac{\partial F_r^l(\boldsymbol{r}(n))}{\partial \boldsymbol{r}(n)}\right|_{\boldsymbol{r}(n)=\boldsymbol{r}_{lpp}} = (\boldsymbol{I} - \boldsymbol{K})DF_r^l(\boldsymbol{r}_{lpp}) + \boldsymbol{K}, \tag{15}$$

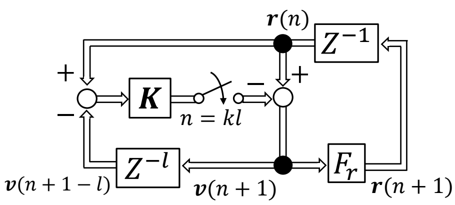

Figure 4: Block diagram of Reservoir Control System for unstable l-periodic orbits detection.

where, $\boldsymbol{r}_{lpp}$ is the reservoir state at l-periodic points and $DF_r^l(\boldsymbol{r}_{lpp})$ in the formula is the Jacobi matrix in $\boldsymbol{r}_{lpp}$ of F_r^l, namely,

$$DF_r^l(\boldsymbol{r}_{lpp}) = \left.\frac{\partial F_r^l(\boldsymbol{r}(n))}{\partial \boldsymbol{r}(n)}\right|_{\boldsymbol{r}(n)=\boldsymbol{r}_{lpp}} = \left.\frac{\partial \boldsymbol{r}(n+l)}{\partial \boldsymbol{r}(n)}\right|_{\boldsymbol{r}(n)=\boldsymbol{r}_{lpp}}. \tag{16}$$

The fixed point of (14) is stable if the absolute value of all eigenvalues of $\boldsymbol{A}_l$ is less than 1. In particular, if the control gain $\boldsymbol{K}$ is set to

$$\boldsymbol{K} = -\left(\boldsymbol{I} - DF_r^l(\boldsymbol{r}_{lpp})\right)^{-1} DF_r^l(\boldsymbol{r}_{lpp}), \tag{17}$$

the fixed point of (14) becomes super-stable because all eigenvalues of $\boldsymbol{A}_l$ are zero.

3. Example of detecting UPOs:Hénon map

Using Hénon map [11] as an example, we use the proposed method to detect UFP and unstable 2-periodic points. The Hénon map is written by

$$\begin{cases} x(n+1) = 1 - 1.4x(n)^2 + y(n) \quad (x(0) = 0.7), \\ y(n+1) = 0.3x(n) \quad (y(0) = 0.21). \end{cases} \tag{18}$$

In this paper, we search for an UFP (x_f) and an unstable 2-periodic point (x_{p1}, x_{p2}) of $x(n)$ of the Hénon map. Figure 5 shows the attractor, x_f and (x_{p1}, x_{p2}) of $x(n)$ of the Hénon map. For the restoration of the Hénon map using RC, the number of training steps is set to $T = 8000$, the input signal sequence $\boldsymbol{U} = [x(100), x(101), \cdots, x(8100)]$ without transient states of the Hénon map and the teacher signal sequence $\boldsymbol{Y}^{teach} = [x(101), x(102), \cdots, x(8101)]$. Other parameters used in this paper are shown in Tab. 1.

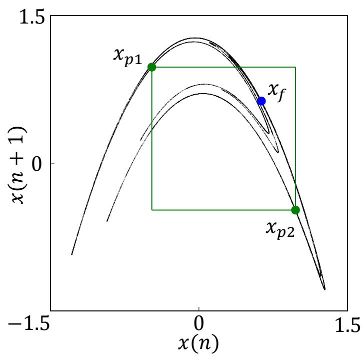

Figure 5: The attractor of $x(n)$ of the Hénon map and the UFP (x_f), the unstable 2-periodic point (x_{p1}, x_{p2}).

Figure 6 shows the UFP and unstable 2-periodic points detected by the proposed method. Figure 6(a) and (b) shows the detected UFP and the time series and attractor of the output $o(n)$ of (6). Figure 6(c) and (d) is the detected unstable 2-period point, and shows the time series and attractor of the output $o(n)$ of (13). The control gain

Table 1: The parameters used in this paper.

Parameters	Value
number of neurons N	500
leak rate ρ	0.8
value of $\boldsymbol{W}^{in}$	binary random with ±0.5
value of $\boldsymbol{W}^{r}$	standard normal distribution
spectral radius of $\boldsymbol{W}^{r}$	0.9999

$\boldsymbol{K}$ are set to (11) and (17), which are super-stable conditions. Detected UPOs are consistent with the UPOs of the Hénon map, confirming that they are converted to stable orbits. Reservoir state at fixed point $\boldsymbol{r}_{fp}$ and Reservoir state at 2-periodic points $\boldsymbol{r}_{2pp}$ are the $\boldsymbol{r}(n+1)$ that satisfied the following Eq. 19 when the restored system (5) was driven.

$$\begin{aligned} &\sum_{i=1}^{N} |r_i(n+1) - r_i(n)| < \epsilon \;\; (\epsilon = 5) \;\; \text{for} \;\; \boldsymbol{r}_{fp}, \\ &\sum_{i=1}^{N} |r_i(n+1) - r_i(n-1)| < \epsilon \;\; (\epsilon = 40) \;\; \text{for} \;\; \boldsymbol{r}_{2pp}. \end{aligned} \tag{19}$$

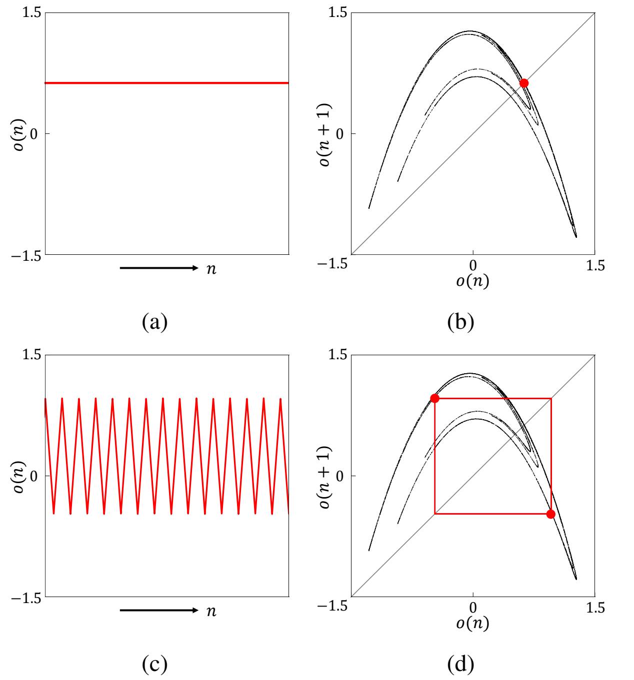

Figure 6: UPOs detected by the proposed method. The black dots are $x(n)$ of the Hénon map. (a)(b) Fixed point, (c)(d) 2-periodic points

4. Conclusion

In this paper, we proposed a new approach for detecting UFP using RC and STM. Furthermore, we show that the proposed method can detect arbitrary periodic UPOs by applying it to fixed point of a composite map. The advantage of proposed method is that it does not require prior chaotic system information or location information of fixed point, and only time series information of the system is used to search for UFP and UPOs. Using the proposed method, we detected UFP and unstable 2-period points of the Hénon map.

One of the future problems is to detect UFP and UPOs in continuous-time and complex systems using the proposed method. In principle, our method can be applied to any system, but it needs to be extended, especially for continuous-time systems.

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