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Numerical Verification of Optimum Point in Linear Programming

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Abstract—In this paper, we are concerned with the following linear programming problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leqq b \text{ and } x \geqq 0,$$

where $A \in \mathbb{F}^{m\times n}$, $b \in \mathbb{F}^m$ and $c, x \in \mathbb{F}^n$. Here, $\mathbb{F}$ is a set of floating point numbers.

The aim of this paper is to propose a numerical method of including an optimum point provided that a good approximation of an optimum point is given.

1. Introduction

In this paper, we are concerned with the following linear programming problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leqq b \text{ and } x \geqq 0, \quad (1)$$

where $A \in \mathbb{F}^{m\times n}$, $b \in \mathbb{F}^m$ and $c, x \in \mathbb{F}^n$. Here, $\mathbb{F}$ is a set of floating point numbers. A dual problem for (1) is given by

$$\text{Minimize } b^t y, \text{ subject to } A^t y \geqq c \text{ and } y \geqq 0, \quad (2)$$

where $y \in \mathbb{F}^m$.

The aim of this paper is to propose a numerical method of including an optimum point of (1) provided that a good approximation of an optimum point is given.

2. Verification Method

In this paper, for two vectors u, v with the same dimension, uv and u/v denote the vectors of the same dimension with components $u_i v_i$ and u_i/v_i, respectively. Then, (1) and (2) are equivalent to the following complimentarity problem

$$f(z) = \begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} = 0 \quad (3)$$

subject to

$$x \geqq 0,\ y \geqq 0,\ b - Ax \geqq 0 \text{ and } A^t y - c \geqq 0, \quad (4)$$

where $z = (x^t, y^t)^t$. The centered path of (3) is defined by

$$f(z) = \begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} = \gamma e, \quad (5)$$

where $e \in \mathbb{R}^{m+n}$ with all elements being 1. The constant γ is defined by

$$\gamma = \frac{\|f(z)\|_1}{m+n} = \frac{(b^t y - c^t x)}{m+n}. \quad (6)$$

Namely, γ is the duality gap of the problem if z is a feasible point.

The Fréchet derivative $f'(z)$ is given by

$$f'(z) = \begin{pmatrix} [A^t y - c] & [x]A^t \\ -[y]A & [b - Ax] \end{pmatrix}, \quad (7)$$

where for a vector $x = (x_1, x_2, \cdots, x_n)^t$, $[x]$ denotes $\mathbf{diag}(x_1, x_2, \cdots, x_n)$. At a given approximate optimum point z, the Newton direction d_n and the centered direction d_c are defined by

$$f'(z) d_n = -\begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} \quad (8)$$

and

$$f'(z) d_c = -\begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} + \gamma e, \quad (9)$$

respectively.

In the method we shall propose, first a feasible point of (3) is searched for a searching direction, which is a linear combination of d_n and d_c, based on the guiding cone method or the penelalized norm method [1]. Here, we assume that we can find an interior point z, which is a good approximation of an optimum point. Then, the second step of the method is to check conditions of the following theorem at the point z:

Theorem 1 *Let $z \in \mathbb{R}^{m+n}$ be an interior point, namely a point satisfying (4) with inequality condition. Let further constants α and ω be defined* *by the ineqalities* $\alpha \geqq \|f'(z)^{-1}\|_\infty \|f(z)\|_\infty$ *and* $\omega \geqq 2(\|A\|_\infty + \|A\|_1)\|f'(z)^{-1}\|_\infty$*, respectively. If*

$$\alpha\omega \leqq \frac{1}{4}, \tag{10}$$

there exists an optimal point $z^* = (x^{*t}, y^{*t})^t \in \mathbb{R}^{m+n}$*, i.e., a point satisfying (3) and (4), enjoying*

$$\|z^* - z\|_\infty \leqq \rho. \tag{11}$$

Here

$$\rho = \frac{1 - \sqrt{1 - 3\alpha\omega}}{\omega}. \tag{12}$$

Before entering proof of Theorem 1, we note that the half assertion of Theorem 1 can be derived from the following Kantorovich theorem for the Newton method:

Theorem 2 (Kantorovich's Theorem) *Let* f *be defined on a ball* $B(z, \hat\rho) = \{\|z' - z\|_\infty \leqq \hat\rho\}$ *with* $z \in \mathbb{R}^{m+n}$ *and* $\hat\rho > 0$*. Let further* $f'(z)$ *be nonsingular and enjoying*

$$\alpha' \geqq \|f'(z)^{-1} f(z)\|_\infty \tag{13}$$

for a certain positive α'*. Furthermore we assume that* f *satisfies*

$$\begin{aligned} & \|f'(z)^{-1}(f'(z') - f'(z''))\|_\infty \\ \leqq \; & \omega' \|z' - z''\|_\infty \text{ for } z', z'' \in B(z, \hat\rho) \end{aligned} \tag{14}$$

with a certain positive constant ω'*. If*

$$\alpha'\omega' \leqq \frac{1}{2}, \tag{15}$$

and

$$\rho' = \frac{1 - \sqrt{1 - 2\alpha'\omega'}}{\omega'} \leqq \hat\rho. \tag{16}$$

there exists a point $z^* = (x^{*t}, y^{*t})^t \in B(z, \rho')$ *satisfying (3). The solution* z^* *of (3) is unique in* $B(z, \rho')$*.*

Proof of Theorem 1 First, we note that f is defined on $\mathbb{R}^{m+n}$. If we put $\alpha' = 1.5\alpha$, then

$$\begin{aligned} \|f'(z)^{-1} f(z)\|_\infty &\leqq \|f'(z)^{-1}\|_\infty \|f(z)\|_\infty \\ &\leqq \alpha \\ &< \alpha'. \end{aligned} \tag{17}$$

Then, it is further noted that

$$\begin{aligned} & f'(z') - f'(z'') \\ = & \begin{pmatrix} [A^t y' - c] & [x'] A^t \\ -[y'] A & [b - Ax'] \end{pmatrix} \\ & - \begin{pmatrix} [A^t y'' - c] & [x''] A^t \\ -[y''] A & [b - Ax''] \end{pmatrix} \\ = & \begin{pmatrix} [A^t(y' - y'')] & [x' - x''] A^t \\ -[y' - y''] A & [b - A(x' - x'')] \end{pmatrix} \end{aligned} \tag{18}$$

It follows from this

$$\begin{aligned} & \|f'(z') - f'(z'')\|_\infty \\ = & \|A^t(y' - y'')\|_\infty + \|(x' - x'')A^t\|_\infty \\ & + \|[y' - y'']A\|_\infty + \|A[x' - x'']\|_\infty \\ = & \|A^t\|_\infty (\|x' - x''\|_\infty + \|y' - y''\|_\infty) \\ & + \|A\|_\infty (\|x' - x''\|_\infty + \|y' - y''\|_\infty) \\ \leqq & 2(\|A\|_\infty + \|A\|_1)\|z' - z''\|_\infty. \end{aligned} \tag{19}$$

Hence, we can use ω in Theorem 1 as ω' in Theorem 2 and

$$\alpha'\omega' = 1.5\alpha\omega \leqq 3/8 < 1/2 \tag{20}$$

holds. Furthermore, ρ' coincides with ρ. Thus, from the Kantorovich theorem (Theorem 2) it is seen that there exists a solution $z^* = (x^{*t}, y^{*t})^t \in B = \{z' \,|\, \|z' - z\|_\infty \leqq \rho\}$ satisfying (3). Further, the Kantorovich theorem states that z^* is unique solution of (3) in the closed ball B.

Next, we show that z^* is feasible, *i.e.*, it satisfies the inequality conditions (4). Let us consider a solution curve of the following continuous Newton method starting from a given feasible point z:

$$\frac{dz(t)}{dt} = -f'(z(t))^{-1} f(z(t)) \text{ with } z(0) = z. \tag{21}$$

The fundamental existence theorem for differential equations states that the solution curve $z(t)$ exists for $t \in [0, M)$ for a certain positive constant M.

Suppose $T \leqq M$ be the smallest value of T such that $z(T)$ is on the boundary of the ball B. Then

$$\|z - z(T)\|_\infty \leqq \int_0^T \left\| \frac{dz(t)}{dt} \right\|_\infty dt < k\|f(z)\|_\infty. \tag{22}$$

Here, k is defined by

$$k = \max_{z' \in B} \|f'(z')^{-1}\|_\infty. \tag{23}$$

This result is derived in [2]. In fact, $z(t)$ satisfies

$$\frac{df(z(t))}{dt} = -f(z(t)) \text{ with } z(0) = z. \tag{24}$$

Thus, $f(z(t)) = f(z)e^{-t}$ holds. Hence, we have

$$\begin{aligned} \left\| \frac{dz(t)}{dt} \right\|_\infty &\leqq \|f'(z(t))^{-1}\|_\infty \|f(z(t))\|_\infty \\ &\leqq k\|f(z)\|_\infty e^{-t}, \end{aligned} \tag{25}$$

which gives

$$\begin{aligned} \int_0^T \left\| \frac{dz(t)}{dt} \right\|_\infty dt &\leqq k\|f(z)\|_\infty (1 - e^{-T}) \\ &< k\|f(z)\|_\infty. \end{aligned} \tag{26}$$

Furthermore, $f(z(t)) = f(z)e^{-t}$ implies $z(t)$ starting with an interior point remains to be an interior point for $t \in [0, M)$.

We note that for $z' \in B$

$$\|f'(z)^{-1}(f'(z) - f'(z'))\|_\infty \leqq \omega\|z - z'\|_\infty \quad (27)$$

holds. We note also that (10) implies

$$\omega\rho < 1. \quad (28)$$

Therefore, from (27), it follows that

$$\begin{aligned} k &= \max_{z'\in B} \|f'(z')^{-1}\|_\infty \\ &\leqq \max_{z'\in B} \frac{\|f'(z)^{-1}\|_\infty}{1 - \|I - f'(z)^{-1}f'(z')\|_\infty} \\ &= \max_{z'\in B} \frac{\|f'(z)^{-1}\|_\infty}{1 - \|f'(z)^{-1}(f'(z) - f'(z'))\|_\infty} \\ &\leqq \max_{z'\in B} \frac{\|f'(z)^{-1}\|_\infty}{1 - \omega\|z - z'\|_\infty} \\ &\leqq \frac{\|f'(z)^{-1}\|_\infty}{1 - \omega\rho}. \end{aligned} \quad (29)$$

Thus, we have

$$\begin{aligned} k\|f(z)\|_\infty &\leqq \frac{\|f'(z)^{-1}\|_\infty\|f(z)\|_\infty}{1 - \omega\rho} \\ &\leqq \frac{\alpha}{1 - \omega\rho}. \end{aligned} \quad (30)$$

Then, we show that

$$\frac{\alpha}{1 - \omega\rho} \leqq \rho \quad (31)$$

holds. In fact, to prove (31) it is enough to show

$$\frac{\alpha}{\sqrt{1 - 3\alpha\omega}} \leqq \frac{1 - \sqrt{1 - 3\alpha\omega}}{\omega} \quad (32)$$

which is equivalent to

$$1 - 2\alpha\omega \leqq \sqrt{1 - 3\alpha\omega}. \quad (33)$$

This is further equivalent to

$$4\alpha\omega \leqq 1 \quad (34)$$

which is now obvious because $\alpha\omega \leqq 1/4$. Thus (31) is shown.

The inequalities (22), (30) and (31) imply

$$\|z - z(T)\|_\infty < \rho \quad (35)$$

which contradicts the fact that $z(T)$ is on the boundary of B. Therefore, there exists no such T and the solution curve is contained in the interior of the ball B. There is no singularity of the right hand side of (21) in B. By the elementary theory of differential equation, the solution can be prolonged to the interval $[0, \infty)$, *i.e.*, $M = \infty$ and it converges to z^* as t tends to ∞. In fact, let z^{**} be a point in the limit set, which is contained in B, of the solution curve. Then z^{**} is a solution of (3). By the uniqueness of the solution of (3) in B, it is identical to z^*. Therfore, the solution curve converges to z^* as t tends to ∞.

Since the solution curve is contained in the feasible set, the limit point z^* is also a feasible point.

(QED)

3. Numerical Example

In this section, let us consider the following simple linear programming problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leqq b \text{ and } x \geqq 0, \quad (36)$$

where $c^t = (300, 300, 500)$,

$$A = \begin{pmatrix} 150 & 100 & 100 \\ 1 & 2 & 1 \\ 0 & 0 & 150 \end{pmatrix} \quad (37)$$

and $b^t = (3000, 40, 1200)$. In this case, we have a feasible solution

$$x = \begin{pmatrix} 5.9999999999999973 \\ 13.000000000000004 \\ 8.0000000000000000 \end{pmatrix}, \quad y = \begin{pmatrix} 1.5000000000000000 \\ 75.000000000000000 \\ 1.8333333333333335 \end{pmatrix}. \quad (38)$$

For this feasible point, we have

$$\alpha\omega < 1.64 \times 10^{-11}. \quad (39)$$

Thus, there exists an optimum solution of (36) in the ball centered at $z = (x^t, y^t)^t$ with a radius

$$\rho = 1.45 \times 10^{-13}. \quad (40)$$

The following is a program of executing verified computation. We have used Scilab on Windows XP with the core two duo Intel processor.

```
format('e',23);
init_round();
c=[300;300;500];b=[3000;40;1200];
A=[150,100,100;1,2,1;0,0,150];
[x,y,h,rho,d,p,pl,dl]=vlinpro(c,A,b)
```

Here, we have used the following function:

```
function [x,y,h,rho,t_d,t_p,pl,dl]=
                                  vlinpro(c,A,b)
// Maximize c^tx, subject to Ax<=b and x>=0
mc=-c;[m,n]=size(A);
q=zeros(n,1);qq=zeros(m,1);
[x,l,f]=linpro(mc,A,b,q,[]);
[y,ly,fy]=linpro(b,-A',-c,qq,[]);
F=[diag(A'*y-c),diag(x)*A';
```

```
                diag(-y)*A,diag(b-A*x)];
r_z=-[x.*(A'*(y)-c);y.*(b-A*x)];
d_n=F\r_z;
ga=1e-14;d_c=F\(r_z+ga*ones(m+n,1));
d=(d_n+d_c)/2;x=x+d(1:n);y=y+d(n+1:m+n);
R=inv(F);
down();
bmAx_l=b-A*x;Aymc_l=A'*(y)-c;
dxA_l=diag(x)*A';dyA_l=diag(-y)*A;
up();
bmAx_u=b-A*x;Aymc_u=A'*(y)-c;
dxA_u=diag(x)*A';dyA_u=diag(-y)*A;
bmAx_c=(bmAx_l+bmAx_u)/2;
Aymc_c=(Aymc_l+Aymc_u)/2;
bmAx_r=bmAx_c-bmAx_l;
Aymc_r=Aymc_c-Aymc_l;
down();
r1_l=x.*Aymc_c-abs(x).*Aymc_r;
r2_l=(-y).*bmAx_c-abs(-y).*bmAx_r;
up();
r1_u=x.*Aymc_c+abs(x).*Aymc_r;
r2_u=(y).*bmAx_c+abs(y).*bmAx_r;
near();
r=[max(abs(r1_l),abs(r1_u));
                max(abs(r2_l),abs(r2_u))];
F_l=[diag(Aymc_l),dxA_l;dyA_l,diag(bmAx_l)];
F_u=[diag(Aymc_u),dxA_u;dyA_u,diag(bmAx_u)];
up();
F_c=(F_l+F_u)/2;F_r=F_c-F_l;
RFmI_u=R*F_c+abs(R)*F_r-eye(m+n,m+n);
down();
RFmI_l=R*F_c-abs(R)*F_r-eye(m+n,m+n);
up();
FF=max(abs(RFmI_l),abs(RFmI_u));
RFmI=norm(FF,'inf');
down();
d=1-RFmI;
up();
niF=norm(R,'inf')/d;nr=norm(r,'inf');
alpha=niF*nr;
omega=2*(norm(A,'inf')+norm(A,1))*niF;
h=alpha*omega;rho=(1-sqrt(1-3*h))/omega;
t_d=b'*y;
down();
t_p=c'*x;pl=b-A*x;dl=A'*y-c;
near();
```

Moreover, rounding modes of double precision floating point numbers are changed by the following functions:

```
function init_round()
     link('up.dll','up','C');
     link('down.dll','down','C');
     link('near.dll','nearest','C');
function up()
   call('up');
function down()
   call('down');
function near()
   call('nearest');
```

Here, the source code of making the near.dll is give by

```
#include <float.h>
unsigned int _controlfp(unsigned int new,
                        unsigned int mask);
void nearest(void) {
   _controlfp(_RC_NEAR,_MCW_RC);
}
```

The source codes for up.dll and down.ll are obtained by replacing the word `_RC_NEAR` by `_RC_UP` and `_RC_DOWN`, respectively.

The following is the result of execution:

```
 dl  =
    0.0000000000000000D+00
    0.0000000000000000D+00
    0.0000000000000000D+00
 pl  =
    4.5474735088646412D-13
    0.0000000000000000D+00
    0.0000000000000000D+00
 p  =
    9.7000000000000000D+03
 d  =
    9.7000000000000018D+03
 rho  =
    1.4429449920498498D-13
 h  =
    1.6306549360861618D-11
 y  =
    1.5000000000000000D+00
    7.5000000000000000D+01
    1.8333333333333335D+00
 x  =
    5.9999999999999973D+00
    1.3000000000000004D+01
    8.0000000000000000D+00
```

References

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