2008 International Symposium on Nonlinear Theory and its Applications
NOLTA'08, Budapest, Hungary, September 7-10, 2008

Analytical BER expression of multi-user chaos-based DS-CDMA system using a piecewise linear chaotic map

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Abstract—In this paper a coherent reception of direct sequence-code division multiple access (DS-CDMA) in multi user case (MU) is considered, when chaotic sequences are used instead of conventional pseudo-noise (PN) spreading codes. A piecewise linear map (PWL) is used as chaotic spreading sequence (CSS). Additive white Gaussian noise channel (AWGN) is assumed. An analytical expression of the bit error rate in the multi user case is presented.

1. Introduction

Many researches have seen high interest in the exploitation of chaos in digital communications [1]. Chaotic signals in communications give more advantages than conventional spread spectrum communication systems, such as robustness in multipath environments, resistance to jamming, low probability of interception [1], etc. Recently, researchers have also studied the multiple access (MA) capability of chaotic signals when they are applied for communication systems [1]. Many coherent chaotic MA systems have been proposed and analyzed [1]. In coherent systems, correlator and threshold detector are used to decode signals, exact replicas of unmodulated chaotic sequences are assumed to be available at the receiver. Theoretical performances of these systems are derived at the receiver, and generally the bit error rate (BER) expressions at the output of the correlators are obtained by mixed analysis-simulation technique [1] [4]. The deterministic nature of chaotic sequences makes that the hypothesis of the independence and the identical distribution (iid) of the chaotic samples is not really acceptable, nevertheless in many cases, theoretical performance is in very good agreement with performance provided by long run Monte-Carlo simulations when iid is assumed. In addition the non periodicity nature of chaotic sequence (CS) makes the constant bit energy assumption not very true and this assumption can lead to inaccurate results. Many papers present the BER expression of chaotic systems by an integral expression, which has not been solved analytically. Numerical integration is the only solution to compute the BER expression. The originality of our paper is to offer an analytical expression of the BER in the case of a piecewise linear map (PWL) chaotic sequence in multi-user case with non constant energy assumption. In our study AWGN channel is considered. Furthermore, we use a coherent chaotic demodulation and perfect synchronisation is assumed. The paper is organized as follows. Second section is dedicated to the transmission system and the multi-user interference. In section 3 the chaotic PWL generator used for spreading data signals is presented. BER integral formulations are established also in this section. Section 4 presents the BER computation methodology. In Section 5 an analytical expression of the BER is derived together with simulation results. Section 6 reports some conclusive remarks.

2. BER formulation

2.1. Transmitter structure of MA chaos-DS-CDMA

We assume that the M users emit with the same power P_e. Same chaotic map is used by all users and different chaotic signals are generated using different initial conditions. Concerning user m, the output of chaotic generator is given by :

$$x^{(m)}(t) = \sum_{k=0}^{\infty} x_k^{(m)} g(t - kT_c) \tag{1}$$

with g is the rectangular pulse of unit amplitude on [0, T_c].

The $x_k^{(m)}$ samples is the output of a discrete time chaos generator process with $x_k^{(m)} = f(x_{k-1}^{(m)})$, and $x_0^{(m)}$ is the initial conditions of chaotic generator of user m. A stream of binary data symbols of user m $S_i^{(m)}$ with period T_s is spreaded by a chaotic signal $x^{(m)}(t)$ at the emitter side. Chaotic sequences are generated independently of one another. Also the mean value of each CS is zeros ($E[x_k] = 0$) to avoid transmitting any noninformation-bearing dc components. Without loss of generality, we will consider those different users are synchronous.
The emitted multiple access signal $e(t)$ is :

$$e(t) = \sum_{n=1}^{M} \sum_{i=0}^{\infty} \sum_{k=0}^{\beta-1} S_i^{(n)} x_{k,i}^{(n)} \tag{2}$$

where $x_{k,i}^{(n)}$ is the k^{th} chip value of the CS used for transmission of the i^{th} symbol by the user n ($x_{k,i}^{(n)} = x_{k+i\beta}^{(n)}$). Parameter β is equal to the number of chaotic samples in a symbol duration ($\beta = T_s/T_c$) and, by analogy with DS-CDMA, we have called this parameter the spreading factor.

2.2. Multi-user interference of MA chaos based DS-CDMA

Principle channel distortion is due to a noise source which is an additive white Gaussian continuous waveform noise (AWGN). At the time decision i, the decision variable at the output of the correlator associated to the user m is :

$$D_i^{(m)} = S_i^{(m)} T_c \sum_{k=0}^{\beta-1} \left(x_{k,i}^{(m)}\right)^2 + T_c \sum_{\substack{n=1 \\ n\neq m}}^{M} \sum_{k=0}^{\beta-1} S_i^{(n)} x_{k,i}^{(n)} x_{k,i}^{(m)} + T_c \sum_{k=0}^{\beta-1} x_{k,i}^{(m)} n_{k,i} \quad (3)$$

where $n_{k,i}$ is the AWGN component during the k^{th} chip of the i^{th} data symbol. This noise has two sides power spectral density given by $N_0/2$. $n_{k,i}$ denotes then a independent random sequence.

$$D_i^{(m)} = S_i^{(m)} T_c \sum_{k=0}^{\beta-1} \left(x_{k,i}^{(m)}\right)^2 + T_c \sum_{k=0}^{\beta-1} x_{k,i}^{(m)} \left(y_{k,i} + n_{k,i}\right) \quad (4)$$

where $y_{k,i} = \sum_{\substack{n=1 \\ n\neq m}}^{M} S_i^{(n)} x_{k,i}^{(n)}$

According to the fact that this quantity is the sum of $M-1$ samples of $M-1$ different zero mean chaotic sequences (from different independent users), we will consider that $y_{k,i}$ is a normal random sequence with the following statistics :

$$E\left[y_{k,i}\right] = 0 \qquad Var\left[y_{k,i}\right] = (M-1)\,\sigma_x^2 \quad (5)$$

where Var denotes the variance, and σ_x^2 is the mean power of a spreading chaotic sequence $x_{k,i}^{(n)}$. It can be also defined that the mean energy of a transmitted chaotic chip of the sequence $x_{k,i}^{(n)}$ is $E_c = T_c \sigma_x^2$.

$n_{k,i}$ and $y_{k,i}$ are then two independent zero mean Gaussian random sequences. At the time decision i, D_i can then be considered as a Normal random variable, with the following moments :

$$E\left[D_i^{(m)}\right] = S_i^{(m)} Eb_i^{(m)} \quad (6)$$

$$Var\left[D_i^{(m)}\right] = Eb_i^{(m)} \left((M-1)\,E_c + N_0/2\right) \quad (7)$$

where $Eb_i^{(m)} = T_c \sum_{k=0}^{\beta-1} \left(x_{k,i}^{(m)}\right)^2$ is the transmitted energy by the user m during its i^{th} data symbol duration.

According to the fact that the data symbol of the user m can be equiprobably on the set $\{+1, -1\}$, and by using (6) and (7), then it comes that the detection probability of symbol i for user m (with energy $Eb_i^{(m)}$) is $P_{er}^{(i,m)}$:

$$P_{er}^{(i,m)} = Q\left(\sqrt{\frac{Eb_i^{(m)}}{(M-1)E_c + N_0/2}}\right) \quad (8)$$

with $Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-u^2}{2}\right)} du$

it appears that the noise performance of the system depends on the energy of the transmitted data symbol by user m. Because the non periodic nature of the chaotic signals, the transmitted bit energy using the chaos based DS-CDMA systems varies from bit to another for the same user m. The total BER will be evaluated by integrating $P_{er}^{(i,m)}$ over all possible values of the energy $Eb_i^{(m)}$ weighted by the probability density function of the energy data symbol $P_{Eb}^{(i,m)}$. The total BER for m user is expressed by :

$$BER_{chaos} = \int_0^{+\infty} Q\left(\sqrt{\frac{\psi}{N_0/2 + (M-1)E_c}}\right) P_{Eb}^{(m)}(\psi)\, d\psi \quad (9)$$

In the practice, the analytical form of this function $P_{Eb}^{(m)}(\psi)$ is not available, but its approximation will be provided in the following section

3. Chaotic generator

One-dimensional noninvertible PWL map is chosen and given by [3]: $\begin{cases} z_k = K\,|x_k| + \phi\,[\mathrm{mod}1] \\ x_{k+1} = sign(x_k)(2z_k - 1) \end{cases}$

This generator depends on parameters K and ϕ. K is a fixed positive integer, ϕ ($0 < \phi < 1$) is a parameter that can be changed to produce different sequences, and the initial condition x_0 will be chosen so that $0 < x_0 < 1/K$. In this paper we consider $K = 3$ and $\phi = 0.1$.

4. BER computation

4.1. Computation methodology of the BER

In order to compute (9) we propose to integrate the following expression :

$$BER_{chaos} = \int_0^{+\infty} Q\left(\sqrt{\frac{2Y^2}{N_0 + 2(M-1)E_c}}\right) p\,(Y)\, dY \quad (10)$$

where $Y = \sqrt{Eb_i^{(m)}}$

This approach appeared to be more tractable because intensive work has been done on the analytical expression resulting of the integration of (10). In the context of mobile radio channels, the BER expression is given by :

$$BER_{\substack{Radio\\channel}} = \int_0^{+\infty} Q\left(\sqrt{\frac{2\lambda^2}{N_0}\xi_b}\right) p(\lambda)\, d\lambda \qquad (11)$$

where λ is the channel gain. Closed form expressions are available for (11) in the case of channels following Rayleigh, Nakagami or Rice distributions. Expression (10) has the same form as (11) and all previous results on (11) can be used for getting an analytical form of integral (10). The choice and the demonstration of this approach can be also found in [6].

4.2. PDF estimation of root square energy

The PDF of the root square energy is fundamental in order to solve (10). The analytical expression of the PDF seems intractable because of the difficulty of solving correlation integral. A second solution is to estimate the root square energy PDF using the histogram, and trying to approximate it by a known PDF. This solution will lead to an analytical BER expression.

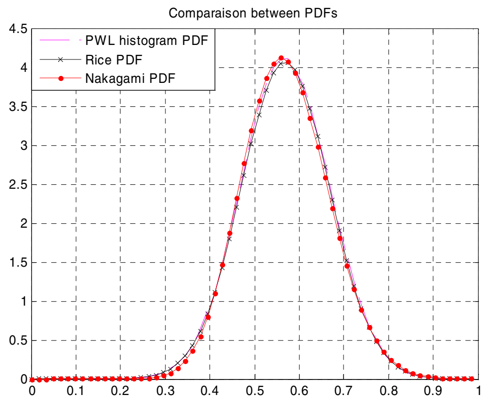

Figure 1: Approximation of the PDF associated to the PWL by a normalized root-square-energy histogram ($K = 3$, $\phi = 0.1$, $\beta = 10$)

The histogram of the root square bit energy obtained for 10^5 samples of the PWL sequence is shown in Figure 1. We have tried to fit it by classical PDF laws. The PDF shape associated to the PWL in Figure 1 gets the possibility to test the two candidates: Rice, and Nakagami PDF. The choice of these two laws has been done because closed analytical forms of (10) are available with these two laws. Figure 1 shows Rice and Nakagami estimated PDF. The Chi-Square Goodness-of-Fit test confirms the fit for the two laws. Nevertheless, Chi-Square test gives priority to the Rice distribution.

5. Analytical BER calculation

This section provides an expression of the BER by approxi-mating the PDF energy with the Rice distribution

5.1. Rice distribution parameters

$R = \sqrt{Eb_i^{(m)}}$ is a random variable following a Rice distribution $P_R(r)$.

$$P_R(r) = \begin{cases} \frac{2(K+1)r}{\Omega} \exp\left(-K - \frac{(K+1)r^2}{\Omega}\right) I_0\left(2\sqrt{\frac{K(K+1)}{\Omega}}r\right) \\ \quad \text{if } r \geq 0 \\ 0 \quad \text{if } r < 0 \end{cases}$$
$$\text{where } r \geq 0, K \geq 0, \Omega \geq 0 \qquad (12)$$

General Rice distribution parameters are given by [5]:

$$\Omega = E(R^2); \qquad \gamma = Var[R^2]/(\Omega)^2 \qquad (13)$$

The shape parameter K can be expressed in terms of γ explicitly as [7]: $K = \frac{\sqrt{1-\gamma}}{1-\sqrt{1-\gamma}}$

5.2. Parameters estimation of the Rice distribution

To have an analytical BER expression, the parameters of the Rice distribution should be derived from the parameters of the PWL sequence. For a normalized CSS power, parameters (13) of Rice distribution are given by :

$$R^2 = Eb_i^{(m)} = T_c \sum_{k=0}^{\beta-1} \left(x_{k,i}^{(m)}\right)^2 \qquad (14)$$

where $x_{i,\beta+k}$ can be seen as a random signal uniformly distributed on the interval $[-1, +1]$ (see [3]). $x_{i,\beta+k}$ has a zero mean and its variance is $1/3$. Then the scale parameter Ω is given by:

$$\Omega = E\left(Eb_i^{(m)}\right) = T_c \sum_{k=0}^{\beta-1} E\left(\left(x_{k,i}^{(m)}\right)^2\right) = \frac{T_c\beta}{3} \qquad (15)$$

Variance of R^2 is then :

$$V[R^2] = E\left(\left(Eb_i^{(m)}\right)^2\right) - \left(\frac{T_c\beta}{3}\right)^2 \qquad (16)$$

with

$$E\left(\left(Eb_i^{(m)}\right)^2\right) = \frac{\beta T_c^2}{5} + 2T_c^2 \sum_{n=1}^{\beta-1} (\beta - n) E\left(x_{k,i}^2 x_{k,i}^2\right) \qquad (17)$$

where $E\left(x_{k,i}^2 x_{i.k+n,i}^2\right)$ is estimated using the PWL chaotic sequence. Then γ can be obtained by (13)

5.3. Analytical expression of the BER

Following results of [5] the analytical BER is given :

$$BER_{PWL}(\beta) = Q(u,v) - \frac{1}{2}\left[1 + \sqrt{\frac{d}{1+d}}\right]\exp\left(-\frac{u^2+v^2}{2}\right)I_0(uv) \quad (18)$$

where

$$u = \sqrt{\frac{\gamma_r^2[1+2d-2\sqrt{d(d+1)}]}{2(1+d)}};\ v = \sqrt{\frac{\gamma_r^2[1+2d+2\sqrt{d(d+1)}]}{2(1+d)}}$$
$$\sigma^2 = \frac{\Omega}{2(K+1)};\quad \alpha^2 = \frac{K\Omega}{(K+1)}$$
$$\gamma_r^2 = \frac{\alpha^2}{2\sigma^2};\ d = \sigma^2\frac{2\xi_b}{N_0+2(M-1)}$$

where ξ_b is the constant bit energy before spreading, and $Q(.,.)$ is the Marcum Q-function [5].
In the simulations we normalized all CS such that $E[x_k^2 = 1]$. In Figure 2, we compare the analytical BER expression of (18), numerical BER computations of (9) using the histogram, and of BER given by Monte Carlo simulations with 10^5 emitted bits. For a lower spreading factor in multi user cases we have also a perfect match between the BER as shown in Figure 2. This result proves the exactitude of our assumption in (5) by considering the MUI as a Gaussian noise.

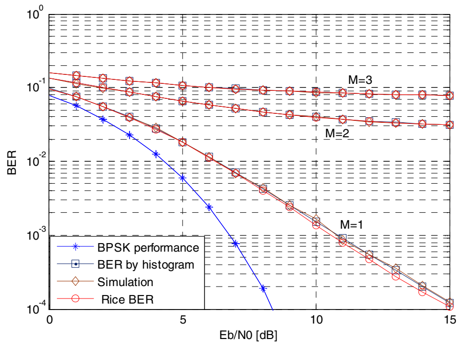

Figure 2: Analytical, Computed, and simulated BER for $\beta = 5$ and $M = 1, 2, 3$ of PWL map with ,$K = 3$, $\phi = 0.1$,

For a DS-CDMA system a low spreading factor has a limited benefit. To improve the BER in chaos-based DS-CDMA we increase the spreading factor. When we increase the spreading factor the variance of the bit energy tends toward zero. This means that the energy distribution in Figure 1 tends to be very sharp around the mean energy when β is high [2]. As shown in Figure 3 the BER in mono user case is very close to the lower BER bound (BPSK case).

6. Conclusion

This paper presents a new methodology for computing the analytical expression of the BER for a PWL sequence in synchronous MA chaos-based DS-CDMA systems. The approximation of the histogram of the root square energy relies on a known PDF distribution (Nakagami or Rice).

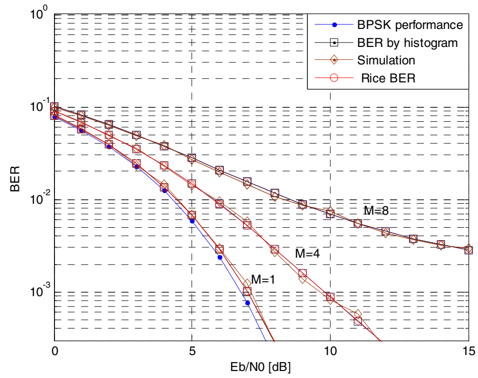

Figure 3: Analytical, Computed, and simulated BER for $\beta = 64$ and $M = 1, 4, 8$ of PWL map with ,$K = 3$, $\phi = 0.1$,

The selection of the best distribution is based on the Chi-square test results. Estimating the histogram of the root square energy by a Rice distribution leads to an analytical expression of the BER. The MUI is computed and approximated by a Gaussian distribution. Simulation results confirm the validity of our analytical expression using Rice approximation and the Gaussian assumption of MUI. Extension of this method to other chaotic maps is under study.

References

[1] F. C. M. Lau and C. K. Tse, "Chaos-Cased Digital Communication Systems," *Springer-Verlag*, Heidelberg, Germany 2003.

[2] G. Kaddoum, P. Chargé, D. Roviras, D. Fournier-Prunaret, "Comparaison of chaotic sequences in a chaos based DS-CDMA system," *NOLTA 2007*, Vancouver, Canada, September 2007.

[3] P. Chargé, D. Fournier-Prunaret, "An M-ary Orthogonal-CSK communication system," *NOLTA 2006*, Bolognia, Italy, Septembre 2006.

[4] A. J. Lawrance and G. Ohama, "Exact calculation of bit error rates in communication systems with chaotic modulation," *IEEE Trans. on circuits and systems-I*, Vol. 50, pp. 1391-1400, 2003.

[5] W. C. Lindsey, "Error probabilities for rician fading multichannel reception of binary and N-ary signals," *IEEE Trans. Inf. Theory*, pp. 339-350, 1964.

[6] G. Kaddoum, D. Roviras, P. Chargé, D. Fournier-Prunaret, "Analytical calculation of BER in communication systems using a piecewise linear chaotic map," *ECCTD 2007*, Seville, Spain, August 2007.

[7] A. Abdi, C. Tepedelenlioglu, M. Kaveh, G. Giannakis, "On estimation of K parameter for Rice fading distribution," *IEEE Commun. Lett.*, vol. 5, pp. 92-94, 2001.