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Phase synchronization in chaotic oscillators induced by common noise

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Abstract—Noise-induced synchronization is a common phenomenon that is often observed from various nonlinear dynamical systems. In this paper, we investigated the noise-induced phase synchronization of chaotic oscillators using the Hilbert transform. We analyzed the noise-induced synchronization in several chaotic oscillators. As a result, we confirmed that single scroll type attractors of a chaotic oscillator can be more easily synchronized than double scroll type attractors.

1. Introduction

Synchronization is a common phenomenon that is often observed in nonlinear dynamical systems[1, 2, 3]. While the synchronization can be caused by mutual couplings, noise-induced synchronization can also be observed theoretically and experimentally in various systems. For example, Mainen and Sejnowski observed a kind of noise-induced synchronization in neural systems. They showed that reliability of occurrences generation improves when neurons of neocortical slices of rats receive noisy inputs[4]. In addition, Neiman and Russell experimentally showed that firing patterns of two uncoupled neurons of paddlefish stochastically synchronized by common noisy currents[5]. These results indicate that common noise induces stochastic synchronization even if oscillators do not have mutual couplings.

In 2004, theoretical results of common white noise synchronization of limit-cycle oscillators were shown by Teramae and Tanaka[6]. In Ref. [6], the phase difference between two oscillators exponentially attenuated by the common white noise. In addition, they analytically showed that Lyapunov exponents predict those obtained by numerical experiments very well.

On the other hand, common noise-induced synchronization of chaotic oscillators has also been investigated. For example, Zhou and Kurths numerically investigated the noise-induced synchronization of chaotic oscillators[7]. Using the Rössler equations[8] and the Lorenz equations[9], they showed that nonidentical chaotic oscillators exhibit phase synchronization if the chaotic oscillator has a common white Gaussian noise.

We have also confirmed that noise-induced phase synchronization of chaotic oscillators exists in real physical systems[10]. In our previous work[10], we used a piecewise linear Rössler circuits[11], the Rössler circuits which we designed and the Lorenz circuits[12]. Although we experimentally show that these oscillators are synchronized by common white noise, we used a different phase definition for the Rössler systems and the Lorenz systems. To investigate basic mechanisms of the common noise synchronization and systematically analyze their characteristic, it is inevitable to compare the results of the Rössler type oscillators and the Lorenz type oscillators, using the same definition of phase differences.

Then, in this paper, we investigated the noise-induced phase synchronization of chaotic oscillators in numerical simulations using the same phase definition. We used the Hilbert transform[13], which enables us to compare the results of different chaotic oscillators. To investigate the common noise-induced synchronization of chaotic oscillators, we used the piecewise linear Rössler oscillators[11], the Lorenz oscillators[12] and the Chua circuit[14]. The Chua circuit has two chaotic attractors: a single scroll attractor and a double scroll attractor. Then, we also examined the noise-induced synchronization in terms of structural difference among chaotic attractors.

In the experiments, we applied the common white Gaussian noise to two uncoupled chaotic oscillators and investigated the frequency distribution of the phase differences between these two chaotic oscillators. We also investigated how chaotic oscillators synchronize in terms of noise intensity. In the real physical systems, we cannot make identical systems, which means that we investigated nonidentical chaotic oscillators. As a result, we found that the Hilbert transform is useful to evaluate how chaotic oscillators synchronize. We also found that synchronizability depends on geometrical shapes of attractors.

2. Chaotic oscillators

2.1. Piecewise linear Rössler oscillator

Heagy et al. investigated the synchronization of the Rössler equations[8] in numerical simulations and experiments[11]. They modified the Rössler equations[8] to implement them by an electrical circuit.

The piecewise Rössler oscillator has three outputs, x, y and z, and is described by the following differential equations

$$\begin{cases} \dfrac{dx}{dt} = -\alpha(\Gamma x + \beta y + \lambda z), \\ \dfrac{dy}{dt} = \alpha(x + \gamma y), \\ \dfrac{dz}{dt} = \alpha[g(x) - z], \end{cases} \tag{1}$$

where the function $g(x)$ is a piecewise linear function, given by

$$g(x) = \begin{cases} 0 & x \leq 3, \\ \mu(x-3) & x > 3, \end{cases} \tag{2}$$

where α, Γ, β, λ and γ are parameters. The parameters in Eq. (1) were set to $\alpha = 1$, $\Gamma = 0.05$, $\beta = 1.0$, $\lambda = 0.5$, $\mu = 15.0$ and $\gamma = 0.133$.

2.2. Lorenz oscillator

We also used the Lorenz system which is implemented in an electric circuit by Cuomo et al[12]. They used a transformation of $\tilde{x} = 2.5 \times 10^3 x$, $\tilde{y} = 2.5 \times 10^3 y$, $\tilde{z} = 2.5 \times 10^3 z$[12]. Then, the Lorenz oscillator is described by the following equations:

$$\begin{cases} \dfrac{d\tilde{x}}{dt} = \sigma(\tilde{y} - \tilde{x}), \\ \dfrac{d\tilde{y}}{dt} = r\tilde{x} - \tilde{y} - 20\tilde{x}\tilde{z}, \\ \dfrac{d\tilde{z}}{dt} = 5\tilde{x}\tilde{y} - b\tilde{z}, \end{cases} \tag{3}$$

The Lorenz system has a geometric shape of double scroll attractors. The parameters in Eq. (3) were set to $\sigma = 16$, $r = 45.8$ and $b = 4.0$.

2.3. The Chua circuit

To investigate the difference of the chaotic attractor produced from the same nonlinear dynamical system, we used the Chua circuit[14]. The Chua circuit also has three outputs x, y and z. The Chua circuit is described by the following differential equations:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{G}{C_1}(y - x) - h(x), \\ \dfrac{dy}{dt} = \dfrac{G}{C_2}(x - y) + z, \\ \dfrac{dz}{dt} = -\dfrac{z}{L}, \end{cases} \tag{4}$$

where the function $h(x)$ is a piecewise linear function, given by

$$h(x) = m_0 x + \frac{1}{2}(m_1 - m_0)(|x + B_p| - |x - B_p|). \tag{5}$$

The parameters in Eqs. (4) and (5) were set to $C_1 = 1/9$, $C_2 = 1$, $L = 1/7$, $m_0 = -0.5$, $m_1 = -0.8$ and $B_p = 1$. To observe a single scroll attractor as shown in Fig. 1(a), we set to $G = 0.649$, and a double scroll attractor as shown in Fig. 1(b), we set to $G = 0.70$.

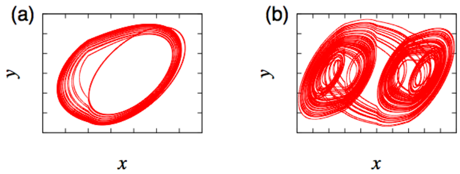

Figure 1: Two attractors observed from the Chua circuit: (a) a single scroll attractor and (b) a double scroll attractor.

3. Method

We defined the phase difference between the outputs of two chaotic oscillators to detect noise-induced synchronization quantitatively. In our previous work[10], we used the phase definition by the Poincaré section and the event-timing-based phase estimation[1]. However, such phase definitions are not enough to evaluate the common noise-induced synchronization if we analyze several chaotic oscillators, because we cannot compare the results of phase synchronization systematically. Then, to compare the phase synchronization in several chaotic oscillators, we use the method based on the Hilbert transform[13].

The Hilbert transform of $x(t)$, $x_H(t)$, and is described by the follwing equation:

$$x_H(t) = \frac{1}{\pi}\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau, \tag{6}$$

where P.V. stands for Cauchy–Principal value. Therefore, we defined the phases $\phi_1(t)$ and $\phi_2(t)$ and phase difference $\theta(t)$ by the following equations:

$$\phi_i(t) = \tan^{-1}\frac{x_{Hi}(t)}{x_i(t)} \quad (i = 1, 2), \tag{7}$$

$$\theta(t) = \tan^{-1}\frac{x_{H1}(t)x_2(t) - x_{H2}(t)x_1(t)}{x_1(t)x_2(t) + x_{H1}(t)x_{H2}(t)}. \tag{8}$$

Using Eqs.(7) and (8), we calculated the frequency distribution of the phase difference $\theta(t)$, and examined whether two chaotic oscillators synchronize by the common white Gaussian noise.

In addition, we changed the noise intensity to examine how chaotic oscillators synchronize in terms of noise intensity. To define the noise intensity, we used signal-to-noise ratio which is described by the following equation,

$$\mathrm{SNR} = 20\log\frac{S}{N}, \tag{9}$$

where S is a variance of the outputs of the chaotic oscillators without noise, and N is a variance of the output of the noise generator.

4. Results

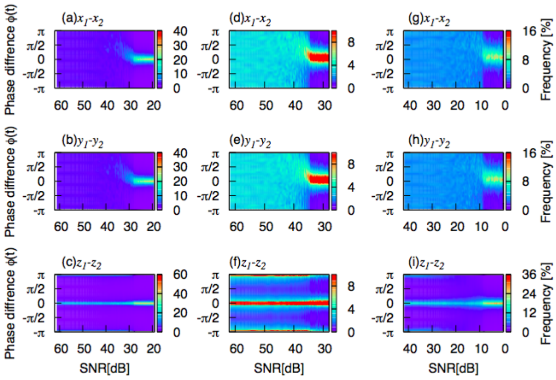

Figure 2: Phase difference of the piecewise linear Rössler oscillators[11]. (a), (b) and (c) are color maps when common noise is applied to the first variable x, (d), (e) and (f) are the case when common noise is applied to the second variable y, and (g), (h) and (i) are the case when common noise is applied to the third variable z. We compared x_1 and x_2 in (a), (d), (g), y_1 and y_2 in (b), (e), (h), and z_1 and z_2 in (c), (f), (i). In these figures, abscissas are the noise intensity defined by Eq. (9) and ordinates are phase difference $\theta(t)$.

Figure 2 shows frequency distributions of the phase difference of $\theta(t)$ between two piecewise linear Rössler oscillators. We observed the phase synchronization of the piecewise linear Rössler oscillators with all variables. Figure

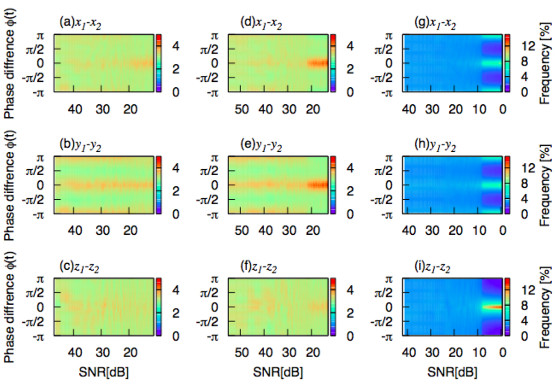

Figure 3: The same as Fig. 2 but for the Lorenz oscillator[12].

3 shows frequency distributions of the phase difference of $\theta(t)$ between two Lorenz oscillators. We observed phase synchronization of the Lorenz oscillators when common white noise is applied to the variables y and z. However, we cannot observe phase synchronization when we applied common noise to the variable x. Figure 4 shows frequency

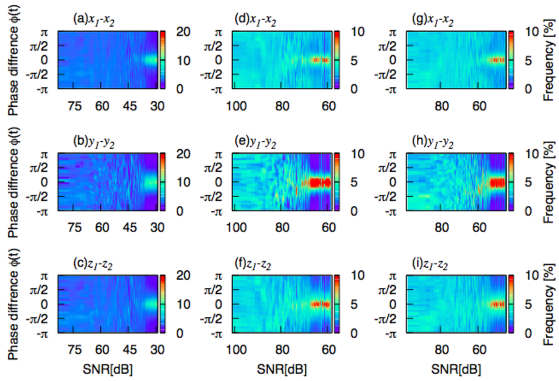

Figure 4: The same as Fig. 2 but for the Chua circuit in case of a single scroll attractor.

distributions of the phase difference of $\theta(t)$ between two Chua circuits that have single scroll attractors. We observed phase synchronization of the Chua circuit wherever we applied common white noise. Figure 5 shows frequency

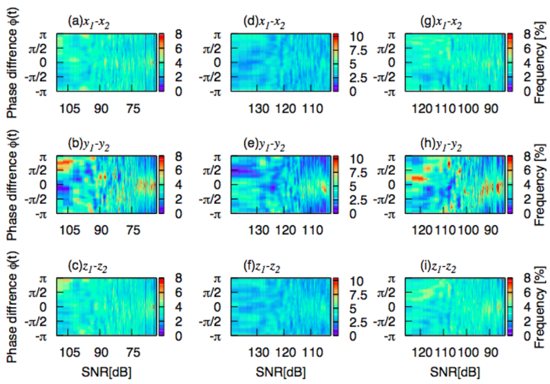

Figure 5: The same as Fig. 4 but the attractor of the Chua circuit in case of a double scroll attractor.

distributions of the phase difference of $\theta(t)$ between two Chua circuits that have double scroll attractors. We observed phase synchronization of the Chua circuit when we applied common noise to the variable y. However, we observed phase synchronization only when we compared y_1 and y_2. To confirm the results of the Chua circuits, we calculated the cross correlation of x_1–x_2, y_1–y_2 and z_1–z_2. Figures 6 and 7 show the cross correlation when we applied common noise to the Chua circuits. We observed that increase of cross correlation corresponded with phase synchronization of the Chua circuit.

For all these results, chaotic oscillators that have a geometric shape of single scroll attractor can be synchronized easily. Chaotic oscillators that have double scroll attractors

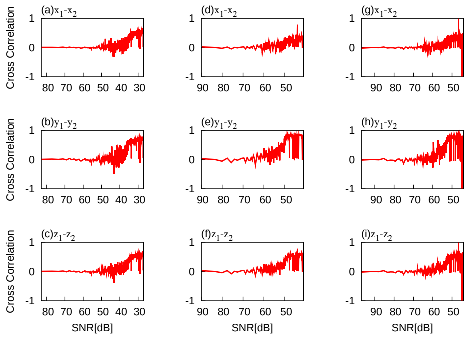

Figure 6: Cross correlation of the Chua circuits in case of a single scroll attractor. We compare the same variables as shown in Fig. 2. In these figures, abscissas are the noise intensity defined by Eq. (9) and ordinates are cross correlation of the variables $x_1 - x_2$, $y_1 - y_2$ and $z_1 - z_2$.

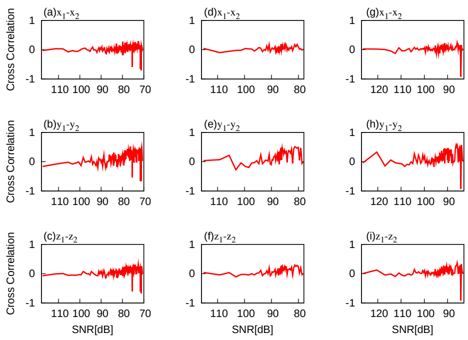

Figure 7: The same as Fig. 6 but for the Chua circuits in case of a double scroll attractor.

can also be synchronized, however, it is relatively harder to synchronize compared with single scroll attractors

5. Conclusions

In this paper, we investigated common noise-induced synchronization of chaotic oscillators. We applied common white noise to two chaotic oscillators and calculated the frequency distribution of the phase difference between two oscillators. We found that single scroll attractors of chaotic oscillators can be synchronized easier than double scroll attractors of chaotic oscillators. As a result, we found that the Hilbert transform is useful to compare how several types of chaotic oscillators synchronize.

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