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Bifurcation analysis of two coupled Izhikevich Oscillators

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Abstract—A simple oscillator of spiking neurons is proposed by Izhikevich. By some numerical experiments, all firing patterns which have been observed in the brain are confirmed at the origin of the mathematical model. Although, a detailed bifurcation analysis has been given by the author, no investigation on its coupling system has been done. In this paper, we consider two Izhikevich neurons coupled by a gap junction. By choosing an appropriate Poincaré section, we can compute bifurcation set for limit cycles. As a result, period-doubling bifurcation and its cascade to chaos is observed by changing the coupling coefficient. We show bifurcation diagrams and numerical simulation results.

1. Introduction

Non-smooth dynamical systems are often derived from actual physical objects, biological activities, electrical systems, and so on. In the mathematical model, there are at least one point which is not differentiable by the state so that it is difficult to treat by using the conventional bifurcation theory. Some theoretical approaches to address this difficulties are available [1, 2], but we also develop a numerical algorithm [3] to maintain non-smooth dynamics by using numerical integration of piecewise differentiable variational equations, i.e., the differential values of the Poincaré mapping required for solving bifurcation problem are evaluated by the solutions of variational equations without any discontinuities. Thus any system including impulses, hysteresis and jumping can be analyzed easily.

A simple oscillator of spiking neurons is proposed by Izhikevich [4]. The biologically plausibility of the oscillator is shown by some numerical experiments. Firing patterns of all known types of neurons are illustrated by changing parameters. The bifurcation sets of this model have been already analyzed [5].

Practical neurons are supposed to be connected any other neurons, however the bifurcation analysis of a junctional Izhikevich oscillator has not been investigated in detail. In this paper, the target is two Izhikevich neuronal oscillators coupled by a gap junction. We investigate bifurcation structures of this models.

2. The method for analysis

If one wants to consider stability or bifurcation problems on the Izhikevich oscillator, calculation becomes complicated because of their non-smoothness. However, analyzing method for the model with nonsmooth function is already provided [3]. It is confirmed that can be calculated by considering switching threshold Poincaré mapping.

Let us consider m-tuple of autonomous differential equations.

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}_k(\boldsymbol{x}, \lambda, \lambda_k), \quad k = 0, 1, \cdots, m-1 \tag{1}$$

where $t \in \boldsymbol{R}$, $\boldsymbol{x} \in \boldsymbol{R}^n$, $\lambda \in \boldsymbol{R}^r$ is an invariant parameter for $\boldsymbol{f}_0, \boldsymbol{f}_1, \cdots, \boldsymbol{f}_{m-1}$ and $\lambda_k \in \boldsymbol{R}^s$ are parameters depending only on $\boldsymbol{f}_k$. r and s are integers. We call these equations piecewise-defined differential equations. Assume that $\boldsymbol{f}_k$ are C^∞-class map for all variables and parameters and every equations in Eq.(1) has a solution with an appropriate initial value $\boldsymbol{x}_{k0}$.

If the orbit is periodic, the following a k-tuple of local mappings are obtained as follows:

$$\begin{aligned}
T_0 : \quad & \Pi_0 \to \Pi_1 \\
& \boldsymbol{x}_0 \mapsto \boldsymbol{x}_1 = \varphi_0(\tau_0, \boldsymbol{x}_0) \\
T_1 : \quad & \Pi_1 \to \Pi_2 \\
& \boldsymbol{x}_1 \mapsto \boldsymbol{x}_2 = \varphi_1(\tau_1, \boldsymbol{x}_1) \\
& \cdots \\
T_{k-1} : \quad & \Pi_{k-1} \to \Pi_0 \\
& \boldsymbol{x}_{k-1} \mapsto \boldsymbol{x}_0 = \varphi_{k-1}(\tau_{k-1}, \boldsymbol{x}_{k-1})
\end{aligned} \tag{2}$$

where $\tau = \sum_{i=0}^{k-1} \tau_i$ is the period of the limit cycle. Then we define the following composite mapping as the Poincaré mapping correlated with Eq.(2).

$$T = T_{m-1} \circ \cdots \circ T_1 \circ T_0 \tag{3}$$

When $\boldsymbol{x}_0$ which started from local section Π_0 passes local section Π_0 again and follow Eq(4), $\boldsymbol{x}_0$ is called a fixed point.

$$\boldsymbol{x}_0 = \boldsymbol{x}_m = T(\boldsymbol{x}_0) \tag{4}$$

The derivative with the initial value of the Poincaré map is given by

$$\left.\frac{\partial T}{\partial \boldsymbol{x}_0}\right|_{t=\tau} = \prod_{k=m-1}^{0} \left.\frac{\partial T_k}{\partial \boldsymbol{x}_k}\right|_{t=\tau_k} \tag{5}$$

Each Jacobian matrix can be written from [3]as follows:

$$\frac{\partial T_k}{\partial \boldsymbol{x}_k} = \left[\boldsymbol{I}_n - \frac{1}{\dfrac{\partial q_{k+1}}{\partial \boldsymbol{x}} \boldsymbol{f}_k|_{t=\tau_k}} \boldsymbol{f}_k|_{t=\tau_k} \frac{\partial q_{k+1}}{\partial \boldsymbol{x}} \right] \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} \tag{6}$$

where I_n is an $n \times n$ identity matrix. $\partial \boldsymbol{\varphi}_k / \partial \boldsymbol{x}_k$ can be obtained by solving the following differential equation:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\right) = \frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\right) \quad \text{with}$$
$$\left.\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\right|_{t=0} = I_n, \quad k = 0,\ 1,\ 2,\ \cdots,\ m-1 \tag{7}$$

Now, we define a local coordinate $\boldsymbol{u} \in \Sigma \subset \boldsymbol{R}^{n-1}$ corresponding to Π_0 by using a projection p and embedding map p^{-1}

$$p^{-1} : \Sigma \to \Pi_0, \qquad p : \Pi_0 \to \Sigma \tag{8}$$

Accordingly, the Poincaré mapping on the local coordinate is obtained as

$$\begin{aligned} T_l : \quad & \Sigma \to \Sigma \\ & \boldsymbol{u} \mapsto p \circ T \circ p^{-1}(\boldsymbol{u}) \end{aligned} \tag{9}$$

The Jacobian matrix is given by

$$\frac{\partial T_l}{\partial \boldsymbol{u}_0} = DT_l(\boldsymbol{u}_0) = \frac{\partial p}{\partial \boldsymbol{x}} \frac{\partial T}{\partial \boldsymbol{x}_0} \frac{\partial p^{-1}}{\partial \boldsymbol{u}} \tag{10}$$

The characteristic equation is given by

$$\chi_l(\mu) = |DT_l - \mu I_{n-1}| = 0 \tag{11}$$

The roots of Eq.(11) $\mu_1, \mu_2, \cdots, \mu_{n-1}$ give multipliers of the fixed points. The derivative with the parameter value of the Poincaré Map is given by

$$\frac{\partial T}{\partial \lambda} = \frac{\partial T_{m-1}}{\partial \boldsymbol{x}_{m-1}} \cdot \frac{\partial T_{m-2}}{\partial \lambda} + \frac{\partial T_{m-1}}{\partial \lambda} \tag{12}$$

Each Jacobian matrix can be written as follows:

$$\frac{\partial T_i}{\partial \lambda} = \frac{\partial T_i}{\partial \boldsymbol{x}_i} \cdot \frac{\partial T_{i-1}}{\partial \lambda} + \frac{\partial T_i}{\partial \lambda} \tag{13}$$
$$\frac{\partial T_0}{\partial \lambda} = \frac{\partial T_0}{\partial \lambda} \tag{14}$$

We used the numerical differentiation as substitute for the second variational because the calculation becomes more complex, and it is enough.

3. Junctional Izhikevich oscillator

A simple oscillator of spiking neurons is proposed by Izhikevich [4]. There are two advantages in this oscillator: first, it does not cost the calculation more than Hodgkin-Huxley-type oscillator. Second, many firing patterns measured in biological activities are reproduced by this model. The equation sets are as follows:

$$\begin{aligned} v' &= 0.04v^2 + 5v + 140 - u + I \\ u' &= a(bv - u) \end{aligned} \tag{15}$$

with the auxiliary after-spike resetting

$$\text{if } v \geq 30\text{mV}, \text{ then } \begin{cases} v \leftarrow c \\ u \leftarrow u + d \end{cases} \tag{16}$$

where, the state variables v and u correspond to the membrane potential of the neuron and membrane recovery variable, respectively. a, b, c, d and I are parameters.

Next, We consider two Izhikevich neurons coupled by a gap junction. The equation set are as follows:

$$\begin{aligned} v_0' &= 0.04v_0^2 + 5v_0 + 140 - u_0 + I + \delta(v_1 - v_0) \\ u_0' &= a(bv_0 - u_0) \\ v_1' &= 0.04v_1^2 + 5v_1 + 140 - u_1 + I + \delta(v_0 - v_1) \\ u_1' &= a(bv_1 - u_1) \end{aligned} \tag{17}$$

with the auxiliary after-spike resetting

$$\begin{aligned} &\text{if } v_0 \geq 30\text{mV}, \text{ then } \begin{cases} v_0 \leftarrow c \\ u_0 \leftarrow u_0 + d \end{cases} \\ &\text{if } v_1 \geq 30\text{mV}, \text{ then } \begin{cases} v_1 \leftarrow c \\ u_1 \leftarrow u_1 + d \end{cases} \end{aligned} \tag{18}$$

where, the parameter δ is coupling coefficient. When $\delta \simeq 1$, each other are connected strongly. By prechecking for this system, not much impressive dynamical behavior with $\delta > 0$, we allow all values for δ without biological assumption.

In the following, we place local section defined by the reset operation:

$$\begin{aligned} \Pi_0 &= \{\boldsymbol{x} = (v_0,\ u_0,\ v_1,\ u_1)^T \in \boldsymbol{R}^4 \mid q_0(\boldsymbol{x}) = v_0 - 30 = 0\} \\ \Pi_1 &= \{\boldsymbol{x} \in \boldsymbol{R}^4 \mid q_1(\boldsymbol{x}) = v_1 - 30 = 0\} \end{aligned} \tag{19}$$

Local mappings are defined as follows:

$$\begin{aligned} T_{00} :\ & \Pi_0 \to \Pi_0 \\ T_{10} :\ & \Pi_1 \to \Pi_0 \\ & v_0 \mapsto c \\ & u_0 \mapsto \varphi_1(t,\ \boldsymbol{x}_0, \boldsymbol{\lambda}) + d \\ & v_1 \mapsto \varphi_2(t,\ \boldsymbol{x}_0, \boldsymbol{\lambda}) \\ & u_1 \mapsto \varphi_3(t,\ \boldsymbol{x}_0, \boldsymbol{\lambda}) \\ T_{01} :\ & \Pi_0 \to \Pi_1 \\ T_{11} :\ & \Pi_1 \to \Pi_1 \\ & v_0 \mapsto \varphi_0(t,\ \boldsymbol{x}_0, \boldsymbol{\lambda}) \\ & u_0 \mapsto \varphi_1(t,\ \boldsymbol{x}_0, \boldsymbol{\lambda}) \\ & v_1 \mapsto c \\ & u_1 \mapsto \varphi_3(t,\ \boldsymbol{x}_0, \boldsymbol{\lambda}) + d \end{aligned} \tag{20}$$

Thus, comparison local mapping is defined as follows:

$$T = T_{10} \circ T_{11} \circ \cdots \circ T_{00} \tag{21}$$

We choose the projection and embedding as follows:

$$\begin{aligned} h :\ & \Pi_0 \to \Sigma \\ & \boldsymbol{x} = (v_0,\ u_0,\ v_1,\ u_1)^T \mapsto \boldsymbol{w} = (u_0,\ v_1,\ u_1)^T \\ h^{-1} :\ & \Sigma \to \Pi_0 \\ & \boldsymbol{w} \mapsto \boldsymbol{x} \end{aligned} \tag{22}$$

4. Bifurcation of fixed point

We compute the bifurcation diagrams in δ–a plane. Figure 1 shows diagrams whose parameters are fixed to $b = 0.2$, $c = -50$, $d = 2$, $I = 10$. A simple Izhikevich oscillator with this parameters shows a period one attractor. However, in the case of two Izhikevich oscillators coupled by a gap junction, the bifurcation phenomena is occurred by changing the parameter δ. In fig.1, the limit cycle bifurcate into chaos attractor by a PD cascade in the right side of chaos region(i). Figure 2 (a) and (b) show a period two and a period four attractors, respectively. Via the PD cascade, a chaotic attractor appears(Fig.2(c)). Figure 3 depicts the one-dimension bifurcation diagram and the maximum Lyapunov exponent value with $a = 0.2$. It is confirmed that a chaos attractor is appeared by a PD cascade.

In $a < 0.173$, a chaos region(ii) exists between period three and period six attractors. The period three attractor disappears by a tangent bifurcation and alternatively a chaotic attractor is shown. Figure 4 shows the one-dimension bifurcation diagram and the maximum Lyapunov exponent value with $a = 0.17$. It is confirmed that a chaos region exists between a period three and a period six regions.

Figure 5 shows bifurcation sets with $b = 0.15$, $c = -50$, $d = 2$, $I = 10$. In this parameter, The bifurcation structure is shape similar to fig.1 but the chaos region(ii) is disappeared. Moreover, in the left side of fig.5, a period one region exists. Only the one neuron spikes with period one in this region. When the switching thresholds are changed, a similar bifurcation structure to fig.1 and 5 are also obtained.

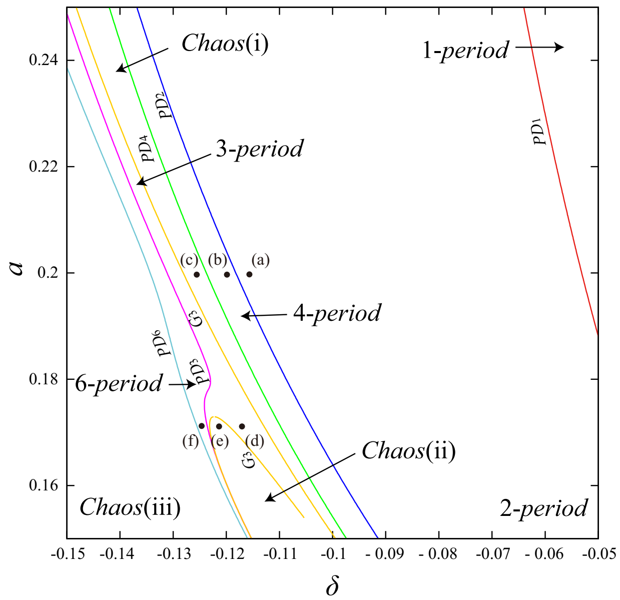

Figure 1: Bifurcation Diagram in the δ–a plane with $b = 0.2$, $c = -50$, $d = 2$, $I = 10$.

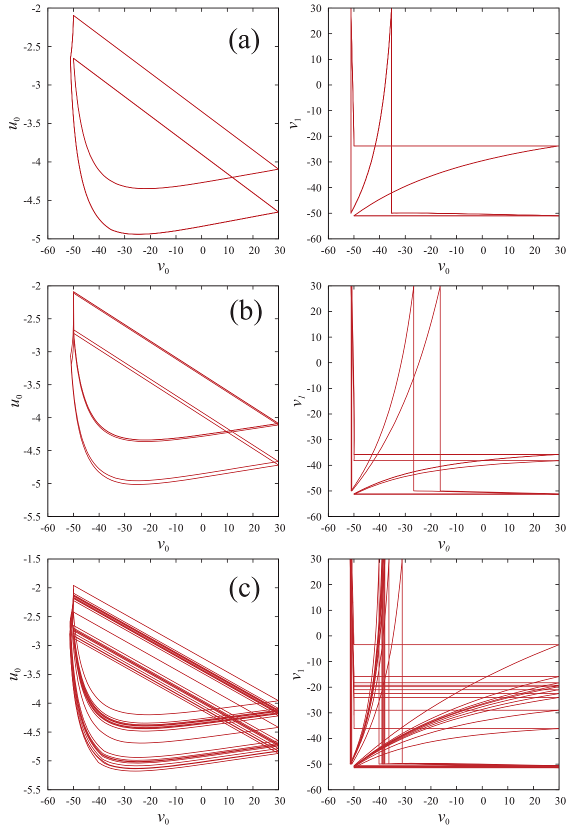

Figure 2: Phase portraits with $a = 0.2$, $b = 0.2$, $c = -50$, $d = 2$, $I = 10$. (a) Period 2, $\delta = -0.115$. (b) Period 4, $\delta = -0.12$. (c) Chaos, $\delta = -0.125$.

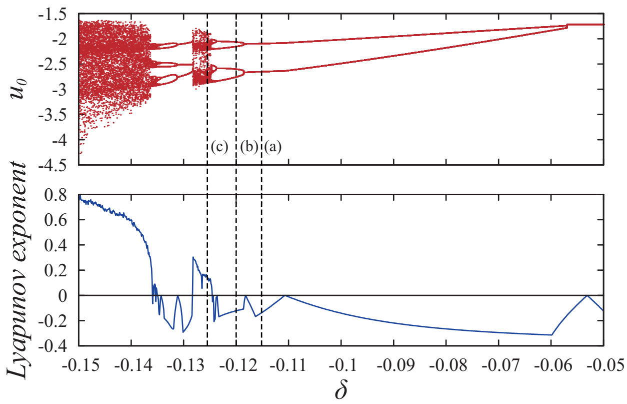

Figure 3: One dimension bifurcation diagram and Lyapunov exponent value with $a = 0.2$, $b = 0.2$, $c = -50$, $d = 2$, $I = 10$.

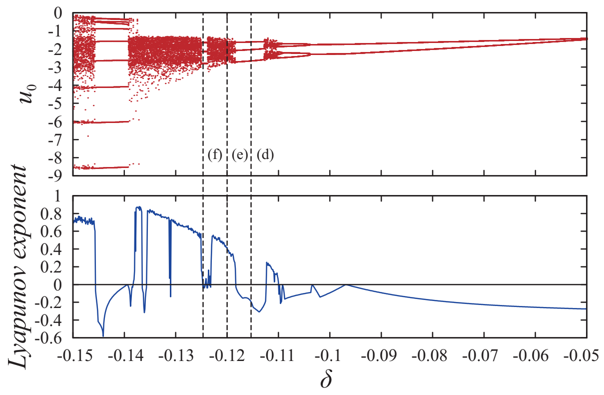

Figure 4: One dimension bifurcation diagram and Lyapunov exponent value with $a = 0.16$, $b = 0.2$, $c = -50$, $d = 2$, $I = 10$.

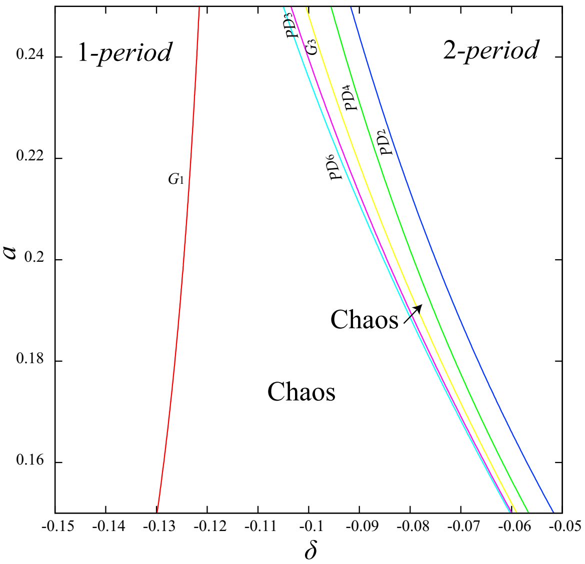

Figure 5: Bifurcation Diagram in the δ–a plane with $b = 0.15$, $c = -50$, $d = 2$, $I = 10$.

Figure 6 shows the bifurcation diagrams in TH_0–TH_1 plane with $a = 0.2$, $b = 0.2$, $c = -50$, $d = 2$, $I = 10$, $\delta = -0.1$. Where, TH_0 and TH_1 are the switching thresholds of each neurons. Bifurcation sets have complex and symmetrical shapes with respect to $TH_0 = TH_1$. It is confirmed that this models are the system with symmetry property.

5. Conclusion

In this paper, we investigated the behavior of two Izhikevich neurons coupled by a gap junction. It was confirmed that this model occurred the bifurcation by coupling coefficient δ. Thus, we computed its bifurcation sets and clarified the bifurcation structure and chaotic regions in δ–a plane.

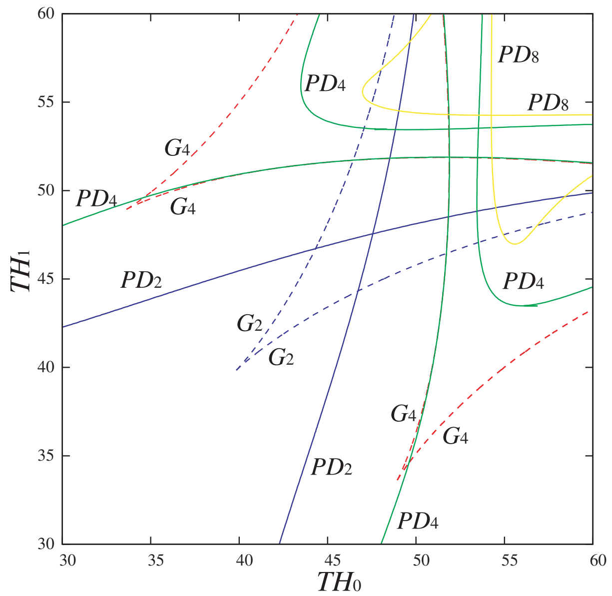

Figure 6: Bifurcation Diagram in the TH_0–TH_1 plane with $a = 0.2$, $b = 0.2$, $c = -50$, $d = 2$, $I = 10$, $\delta = -0.1$

In the future, we want to compute bifurcation sets at other parameters and investigate the model connected more neurons.

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