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Spectral analysis of the propagating pulse wave in 6 coupled bistable oscillators

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Abstract—Frequency components of the propagating pulse wave observed in 6 coupled bistable oscillators are investigated. It is demonstrated that the propagating pulse wave shows a multimode oscillation. These frequency components are clearly changed, as the degree of nonlinearity becomes weak. For weak nonlinear case, the spectrum of the solution shows four dominant peaks, which attracts controversy based on the past theoretical works.

1. Introduction

It is frequently observed in various fields that coupled multiple elements which have autonomous dynamics become to have a valuable function as a unit. Propagating wave phenomenon, which may be utilized modeling of biological information processing[1], corresponds to one of them. Various propagating wave phenomena in several systems such as chaotic pulse, propagation of phase states, etc. have been investigated and have attracted constant interest in various areas[2, 3].

In our previous work, we demonstrated that there exists the propagating pulse wave in an inductor-coupled bistable oscillator array even in a practical setting[4]. The propagating pulse wave consists of several adjacent oscillators oscillating with large amplitude, and the part of large amplitude oscillation in the array propagates with a constant speed.

In the meanwhile, to provide a theoretical framework for understanding this system, the analysis of oscillation modes based on the averaging method or the perturbation method for weakly nonlinear case were extensively performed in [5, 6]. One of the notable results is an out-of-phase synchronization of envelopes (the synchronized multimode waves)[6]. This solution can be considered as a kind of propagating wave phenomena, and seems to be similar to the propagating pulse wave in our case. In order to make understanding of the propagating pulse wave forward, the differences between the results in our case and the similar solution should be appreciated.

The propagating pulse wave shows an almost periodic oscillation. In order to demonstrate the property, it is necessary to investigate the dominant oscillation frequencies. In addition, because the dominant frequency of the solution in [6] agrees well with the theoretical value, we can draw a comparison between two solutions for weak nonlinearity.

In this paper, we investigate spectral property of the propagating pulse wave observed in 6 coupled bistable oscillators. Firstly, we show that how frequency components of the propagating pulse wave are included for strong nonlinear case. Then, we pay attention to how its frequency components are changed as the degree of nonlinearity becomes weak. In addition, we compare the results with the similar solution, and discuss the differences between them.

2. Fundamental equation

The system equation of a ring of inductor-coupled bistable oscillators can be written in the following with reference to [4] :

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{y} \\ \dot{\mathbf{y}} &= -\varepsilon(1-\beta\mathbf{x}_C + \mathbf{x}_f)\mathbf{y} - \mathbf{K}_N\mathbf{x} \\ (\cdot &= d/dt), \end{aligned} \qquad (1)$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_N]^T$, $\mathbf{y} = [y_1, y_2, \cdots, y_N]^T$, $\mathbf{x}_C = [x_1^2, x_2^2, \cdots, x_N^2]^T$, $\mathbf{x}_f = [x_1^4, x_2^4, \cdots, x_N^4]^T$, and

$$\mathbf{K}_N = \begin{bmatrix} 1+\alpha & -\alpha & & & -\alpha \\ -\alpha & 1+\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & & \ddots & \ddots & -\alpha \\ -\alpha & & & -\alpha & 1+\alpha \end{bmatrix},$$

N is the number of oscillators. The $x_i, i = 1, 2, \cdots, N$ denotes the normalized output voltage of the i-th oscillator, y_i denotes its derivative. The parameter ε (> 0) shows the degree of nonlinearity. The parameter α ($0 \leqq \alpha \leqq 1$) is a coupling factor; namely $\alpha = 1$ means maximum coupling, and $\alpha = 0$ means no coupling. The parameter β controls amplitude of oscillation. Each isolated oscillator has two steady-states, namely, no oscillation and periodic oscillation depending on the initial condition. Hereafter, we investigate spectral property for $N = 6$ case, and the parameter β is fixed to 3.2 for simplicity.

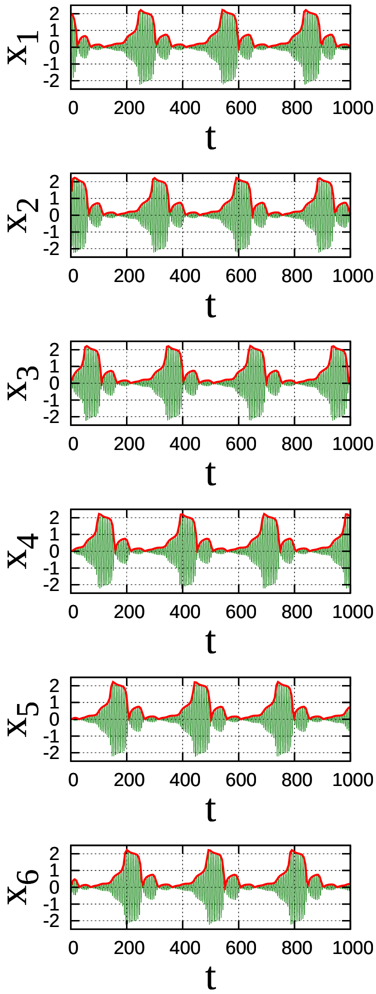

Figure 1: A typical propagating pulse wave for strong nonlinear case. Time waveforms (the associated upper envelopes) are indicated with green (red) curves. The initial condition is given as $x_1 = 2.0, y_2 = 1.3$ and all other variables are zero. ($\varepsilon = 0.36, \alpha = 0.1$ and $\beta = 3.2$)

3. Propagating pulse wave and its frequency components

There exists the propagating pulse wave in the system (1). In the following, we review the solution for strong nonlinear case, and then we investigate its spectral property as a first step. Next, frequency component variations with respect to the degree of nonlinearity is investigated. Then, we will discuss the results based on the perturbation method, and compare with the similar solution for weak nonlinear case.

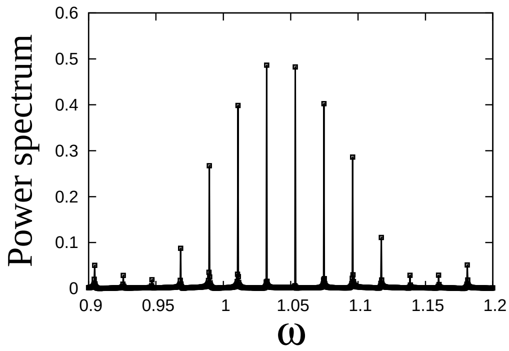

Figure 2: The power spectrum of x_1 in Fig.1.

3.1. Strong nonlinear case

A typical propagating pulse wave for $\varepsilon = 0.36$ and $\alpha = 0.1$ is depicted in Fig.1[1]. It is clear that the part of large amplitude oscillation in the array propagates with a constant speed. It should be noted that the solution shows an almost periodic oscillation. The corresponding envelope appears in the form of almost periodic oscillation as a corollary.

To detect the dominant component frequencies of x_i, $i = 1, 2, \cdots, 6$, Fast-Fourier Transform (FFT) is applied. The power spectrum of x_1 associated with the propagating pulse wave in Fig.1 is illustrated in Fig.2[2]. The solution has several dominant peaks, namely this is a multimode oscillation. In the following section, variations of these frequency components with decreasing ε are investigated.

3.2. Variations with the degree of nonlinearity

The propagating pulse wave appears via some kind of global bifurcation[7]. The bifurcation occurs at the point around the pitch-fork(PF) bifurcation point (α_{PF}) of a certain kind of periodic solution. Figure 3 shows a PF bifurcation set of the periodic solution in the $\varepsilon - \alpha$ plane. In neighborhood of the right side of the bifurcation set, the propagating pulse wave appears. That is, at least for $0.01 \leqq \varepsilon \leqq 0.36$, there exists the propagating pulse wave near the bifurcation set, which can be seen by direct computer simulation of Eq.(1).

It is depicted in Fig.4 that the frequency components of the propagating pulse wave are changed with the value of ε. The dominant frequency components are indicated by the sharp line. As an example, Fig.2 shows nearly comparable result for $\varepsilon = 0.36$[3]. It should be noted that all dominant frequency components approach to $\omega = 1.0$ rad/sec, as ε becomes small.

[1] All numerical integrations are carried out by 4th order Runge-Kutta method with a step size of 0.01.

[2] Power spectra in this paper are obtained from 2^{21} sampling points with 48μHz resolution.

[3] To be exact, there is a slight difference with the value of α. $\alpha = 0.1$ in Fig.2, and $\alpha = 0.101577$ in Fig.4 for $\varepsilon = 0.36$. The reason is found in the caption in Fig.4.

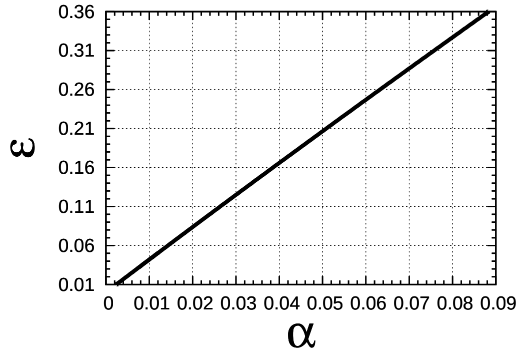

Figure 3: Pitch-fork bifurcation set in the $\varepsilon - \alpha$ plane of the periodic solutions obtained from the initial condition $x_i = 2.0, i = 1, \cdots, 6$ and all other state variables being zero.

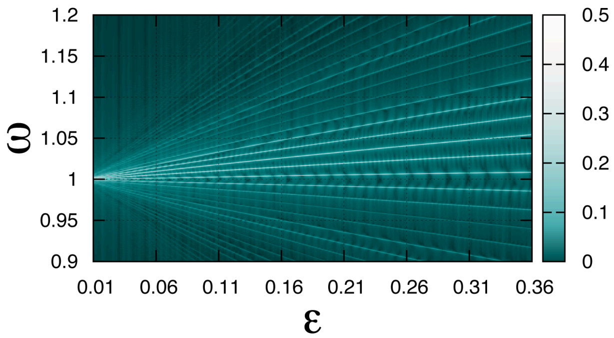

Figure 4: Variations of frequency components of the propagating pulse wave in terms of ε. α is set to $1.15 \times \alpha_{PF}$ so that the solution can exist. The brighter part indicates the stronger frequency component.

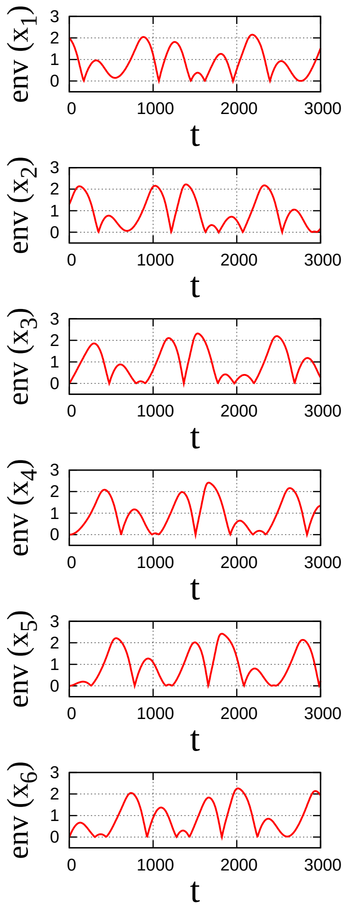

Figure 5: The envelopes of x_i for $i = 1, \cdots, 6$ of the propagating pulse wave obtained from the same initial condition in Fig.1 for $\varepsilon = 0.01, \alpha = 0.01$ and $\beta = 3.2$.

3.3. Weak nonlinear case

According to [6], the multimode (M modes, $M = 1, 2, \cdots$) waves can occur for weak nonlinear case. These stabilities entirely depend on the values of M and β ($-g_3$ in the literature). For $\beta = 3.2$, the stabilities of M modes oscillations are calculated in Table 1 referring to the literature, where $r^0_{i,m}$ $(i, m = 1, \cdots, M)$ correspond to the approximates of stationary amplitudes of equilibrium point in the perturbation method, and the corresponding eigenvalues decide the stability of each equilibrium point. It should be noted that it is indicated that the solutions for $M = 1, 2$ (with the bold strokes in Table.1) exists stably. On the other hand, there is no solution which is stable in this case for $3 \leqq M \leqq 100$ (we only show the results up to $M = 5$ because of space limitations).

In [6], the quasi-periodic oscillations are investigated, some of which (especially for $M = 2$) have very interesting synchronous property on envelope of each time waveform. Then, we investigate what kind of multimode oscillation (almost periodic oscillation) is resulted when ε becomes small enough so that the perturbation method is effective.

Figure 5 shows an upper envelopes (env $(x_i) > 0$) of the time waveforms of x_i, $i = 1, \cdots, 6$ for $\varepsilon = 0.01$ and $\alpha = 0.01$. Being different from [6], the shape of these envelopes are different, and simple phase synchronization of them cannot be recognized. That is, the envelopes appears in the form of almost periodic oscillation similarly to them for strong nonlinear case. The power spectrum of x_1 for this case is depicted in Fig.6. The spectrum shows four dominant peaks, therefore, this is a four-fold multimode oscillation.

Interestingly, the dominant peaks are nearly equal to angular frequencies of resonance modes (ω_i, $i = 1, \cdots, N$) with $\varepsilon = 0$ in Eq.(1). ω_i can be explicitly expressed by the square root of the eigenvalues of $\mathbf{K}_N$ in Eq.(1) in the following[6].

$$\omega_i = \sqrt{1 + \alpha - 2\alpha \cos \frac{2\pi(i-1)}{N}} \qquad (2)$$

Then, the number of different modes becomes $1 + [N/2]$ ($[\cdot]$ is Gauss's symbol). Therefore, for $N = 6$ there are

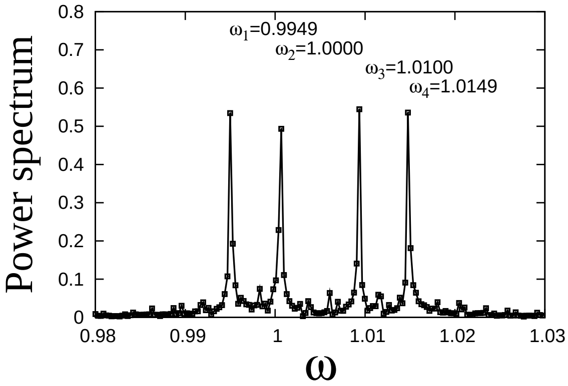

Figure 6: Power spectrum of the same solution in Fig.5. In addition, $\omega_1, \omega_2 (= \omega_6), \omega_3 (= \omega_5)$ and ω_4 in Eq.(2) are shown.

Table 1: Equilibrium points and their stabilities for $\beta = 3.2$ obtained by the perturbation method[6]. The bracketed number indicates the multiplicity of eigenvalues.

M	Equilibrium points	Eigenvalues
1	$r_1^0 = 1.305$	0.637
	$\mathbf{r_1^0 = 2.167}$	**-1.757**
2	$r_1^0 = r_2^0 = 0.782$	0.533, -0.302
	$\mathbf{r_1^0 = r_2^0 = 1.144}$	**-0.191, -1.141**
	$r_1^0 = 1.673, r_2^0 = 0.632$	0.287, -1.407
3	$r_i^0 = 1.575, r_m^0 = 0.424$	0.221 (2), -1.112
4	$r_i^0 = 1.550, r_m^0 = 0.338$	0.192 (3), -1.124
5	$r_i^0 = 1.539, r_m^0 = 0.288$	0.174 (4), -1.186

four different angular frequencies ($\omega_1, \omega_2, \omega_3$ and ω_4). For $\alpha = 0.01$ these values are shown in Fig.6, which correspond to the four dominant frequencies of the propagating pulse wave for $\varepsilon = 0.01$. The result may suggest that theoretical analysis can be applied to this kind of propagating wave.

4. Conclusions

It is demonstrated that the propagating pulse wave shows the multimode oscillation. These frequency components approach to 1.0 rad/sec, as the degree of nonlinearity becomes weak. For weak nonlinear case, the spectrum of the solution shows four dominant peaks. In this meaning, this is a different type of multimode oscillation (especially for $M = 2$) from those in [6]. In while, the dominant peaks agree with ω_i in Eq.(2), which is remained to be investigated in detail in the future.

Acknowledgments

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References

[1] J.P. Keener, “Propagation and its failure in coupled systems of discrete excitable cells, ” SIAM J. Appl. Biol., vol.47, no.3, pp.556-572, 1987.

[2] Y. Nishiura, D. Ueyama and T. Yanagita, “Chaotic pulses for discrete reaction diffusion systems”, SIAM J. Applied Dynamical Systems, vol.4, no.3, pp.723-754, 2005.

[3] M. Yamauchi, M. Wada, Y. Nishio and A. Ushida, “Wave propagation phenomena of phase states in oscillators coupled by inductors as a ladder”, IEICE Trans.Fundamentals, vol.E82-A, no.11, pp.2592-2598, 1999.

[4] K.Shimizu, T. Endo and D.Ueyama, “Pulse wave propagation in a large number of coupled bistable oscillators”, IEICE Trans. Fundamentals, vol.E91-A, no.9, pp.2540-2545, 2008.

[5] T. Endo and T. Ohta, “Multimode oscillations in a coupled oscillator system with fifth-power nonlinear characteristics”, IEEE Trans. on circuit and systems, vol.cas-27, no.4, pp.277-283, 1980.

[6] T. Yoshinaga and H. Kawakami, “Synchronized quasi-periodic oscillations in a ring of coupled oscillators with hard characteristics”, Electronics and Communications in Japan, Part III, vol.76, no.5, pp.110-120, 1993.

[7] K.Shimizu, M.Komuro and T.Endo, “ Bifurcation analysis of the transition from the standing pulse wave to the propagating pulse wave in a ring of coupled bistable oscillators ”, International Symposium on Nonlinear Theory and its Applications (NOLTA 2009), pp.171-174, Sapporo, Japan, Sep. 2009.