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Complex Time Homotopy for Finding Periodic Oscillations

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Abstract—This paper proposes to extend the time variable of circuit equations to the complex number field for finding out periodic oscillations in global parameter space. The extension reveals the reason why the homotopy path of complex state variables breaks. Further, the singularities on the complex time plane lead to the classification of the periodic oscillation. The classification enables to search for real solutions efficiently.

1. Introduction

In nonlinear circuit analysis, it is important to find out circuit parameters on which target oscillations are generated. Once we find out the parameters, we can apply efficient tools, e.g., homotopy method which have global convergence [1] and interval method which enables to calculate all solutions of nonlinear equations [2, 3, 4]. However, in order to find out the circuit parameters, we have no efficient tools, i.e., we have to solve determining equations of the target oscillations a number of times in global parameter spaces through a trial and error process.

Recently, in order to systematically find out circuit parameters of periodic oscillations, homotopy with complex state variables has been proposed [5]. The complex state variables enable to find out the complex periodic solutions at any circuit parameters even if the real solution does not exist. Further, the analyticity of the complex function represented by Cauchy-Riemann equations derives the monotonic homotopy path with respect to the homotopy parameters [1]. Using the monotonicity of general homotopy path, we can systematically find out real periodic solutions in the global parameter spaces.

Although the homotopy with the complex state variables is a vital tool for the systematic search, the homotopy path breaks in some cases. This paper reveals the reason why the homotopy path breaks by extending the time variable to the complex number field. Then, the complex time homotopy leads to classification of the periodic oscillations and we propose to use the classification for efficiently finding out real periodic solutions.

In section 2, we review the method in [5] and show an example of the break of the homotopy path. In section 3, we introduce the complex time variable for the circuit equation and reveal that movable singularities on the complex time plane break the homotopy path. In section 4, we propose to classify the oscillations using the movable singularity. In section 5, we show that the classification is efficient for the systematical search for real periodic oscillations in the global parameter spaces.

2. Homotopy with Complex State Variable

2.1. Determining equation of periodic oscillation

First, we review the method for finding periodic oscillations using the homotopy with the complex state variables [5]. We consider the following scaled real circuit equation;

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{e}(t) \tag{1}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)' \in \boldsymbol{R}^n$ is a real vector of state variables and the prime means transpose, and $\boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{R}^n \mapsto \boldsymbol{R}^n$. The vector $\boldsymbol{e}(t) \in \boldsymbol{R}^n$ corresponds to AC sources of period 2π:

$$\boldsymbol{e}(t + 2\pi) = \boldsymbol{e}(t). \tag{2}$$

By extending the real state variables to the complex number field, the Eq.(1) is redefined by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{f}(z) + \boldsymbol{e}(t). \tag{3}$$

where $z \in \boldsymbol{C}^n$ and we assume that $\boldsymbol{f}(z) : \boldsymbol{C}^n \mapsto \boldsymbol{C}^n$ is analytic. The time t, the source $\boldsymbol{e}(t)$ and the coefficients of the polynomials are still in real number field.

We derive the determining equation of the periodic solution of Eq.(3) by shooting method [6]. The integration of Eq.(3) from an initial value $z(0) = z_0$ gives

$$z(t) = z_0 + \int_0^t \boldsymbol{f}(z) + \boldsymbol{e}(s)\mathrm{d}s. \tag{4}$$

A problem of finding a periodic solution of period $T \in \boldsymbol{R}$ is a two-point boundary value problem in which the solution of Eq.(3) in the interval $[0, T]$ must satisfy the boundary condition $z(0) = z(T)$. Assuming that we can integrate Eq.(3) from $t = 0$ to $t = T$, we express the above problem using a mapping $\boldsymbol{T} : \boldsymbol{C}^n \mapsto \boldsymbol{C}^n$,

$$z_0 = \boldsymbol{T}(z_0), \quad \boldsymbol{T}(z_0) \equiv \int_0^T \boldsymbol{f}(z) + \boldsymbol{e}(s)\mathrm{d}s + z_0. \tag{5}$$

Thus, the determining equation of the periodic oscillation is defined by

$$\boldsymbol{F}(z_0) \equiv z_0 - \boldsymbol{T}(z_0) = \boldsymbol{0}. \tag{6}$$

2.2. Algorithm for finding out periodic solutions

In order to solve the Eq.(6), we apply Newton homotopy which has more global convergence than Newton method [1]. The analyticy of Eq.(6) makes the homotopy path monotonic with respect to the homotopy parameter [5] and we obtain the complex periodic solutions of Eq.(6). From the complex periodic solutions, we search for real periodic solutions using the general homotopy which also has monotonic paths with respect to the homotopy parameter [5]. The monotonicity enables the systematic search for the real periodic solutions in the global parameter space. The algorithm for finding out real periodic solutions are summarized by the following procedure:

Step 1: We fix circuit parameters.

Step 2: We give enough initial vectors $\hat{\boldsymbol{a}}$ and find all or almost all complex periodic solutions of Eq.(6) using the Newton homotopy.

Step 3: We find out real periodic solutions of Eq.(6) from the complex solutions obtained in Step 2 using the general homotopy. The monotonicity of the general homotopy enables to search for the real solutions in the global parameter space.

If we obtain enough solutions in the Step 2, we do not need to execute the Newton homotopy for other values of the circuit parameters.

2.3. Break of homotopy path

Although the algorithm gives the systematic search by the monotonic homotopy path, the homotopy path breaks in some cases. For example, let us consider a RLC resonance circuit shown in Fig.1. The scaled circuit equation is

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = -z_2 - \zeta I(z_1) + E\sin(t)$$
$$\frac{\mathrm{d}z_2}{\mathrm{d}t} = \eta I(z_1), \qquad (7)$$

where z_1 and z_2 correspond to the flux interlinkage of inductor and the capacitor voltage respectively, and ζ and η correspond to resistance and capacitive susceptance respectively. The magnetizing characteristics of nonlinear inductor is approximated by $I(\Psi) = \Psi^3$. At the first step of the algorithm, we give a set of circuit parameters by $(E, \xi, \eta) = (0.5, 0.15, 0.4)$. Second, we apply the Newton homotopy to Eq.(6) and obtain 5 solutions $S_0, \ldots, S_4$. Third, we apply the general homotopy from the 5 solutions and obtain the homotopy paths shown in Fig.2. The figure shows the real part of $z_1(0)$ and the real solutions are denoted by solid line. The branch AB which is traced from S_1 is broken at the point B. The waveform on B is shown in Fig.3. The impulse-like waveform indicates that a movable singularity of Eq.(7) breaks the homotopy path. In order to confirm the fact, we extend the time variables to the complex number field.

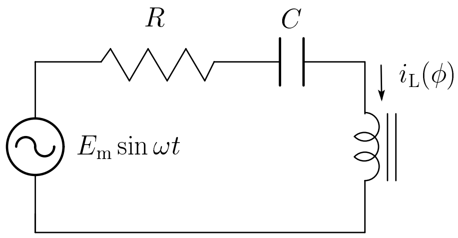

Figure 1: RLC resonance circuit with nonlinear inductor

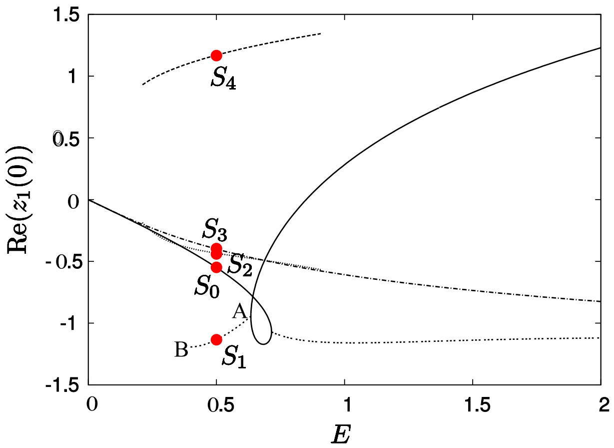

Figure 2: Homotopy paths traced from $S_0, \ldots S_4$.

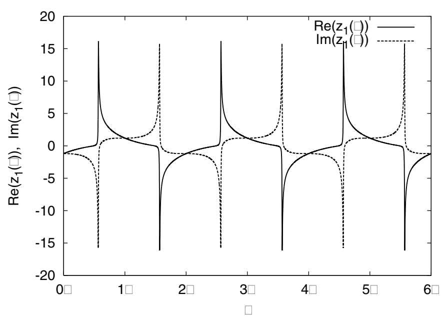

Figure 3: Periodic solution on the break point B

3. Waveform on Complex Time Plane

We extend the time variable of the circuit equation (3) to the complex number field as

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \boldsymbol{f}(z) + \boldsymbol{e}(\tau), \qquad (8)$$

where $\tau \in \boldsymbol{C}$. This extension gives the freedom of integration path on the complex time plane and we redefine the integration (4) by

$$z(t) = z_0 + \int_\gamma \boldsymbol{f}(z) + \boldsymbol{e}(s)\mathrm{d}s, \qquad (9)$$

where γ denotes the integration path. We redefine the mapping $\boldsymbol{T}(z_0)$ in Eq.(5) by the integration path from $\tau = 0 + 0\mathrm{i}$ to $\tau = T + 0\mathrm{i}$ on the complex time plane where i denotes $\sqrt{-1}$. The value $z(T)$ depends on the relative position of the integration path and movable singularities on the complex time plane. Figure 4 shows the case that two integration paths $\gamma^{(1)}$ and $\gamma^{(2)}$ gives different $z(T)$. That is, If there exists movable singularity in the closed contour $\gamma^{(1)}$ and $\gamma^{(2)}$, the values $z(T)$ by $\gamma^{(1)}$ and $\gamma^{(2)}$ are different in general [7].

In order to confirm the existence of the singularity for the waveform in Fig.3, we calculate the waveform $z(\tau)$ on the complex time plane shown in Fig.5. The figure indicates that the impulses on the complex time plane are movable branch points of $z(\tau)$ [7].

Figure 6 illustrates the mechanism of the break of the homotopy path. When we are tracing the general homotopy path, if a movable branch point crosses the real axis on the complex time plane, the $z(T)$ for the real integration paths jumps to the different point on the Riemann surface of $z(\tau)$ and the homotopy path is broken.

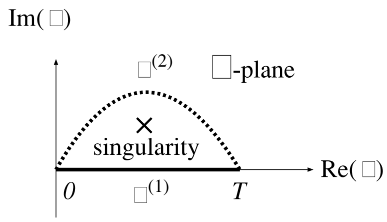

Figure 4: Example of two integration path on τ-plane.

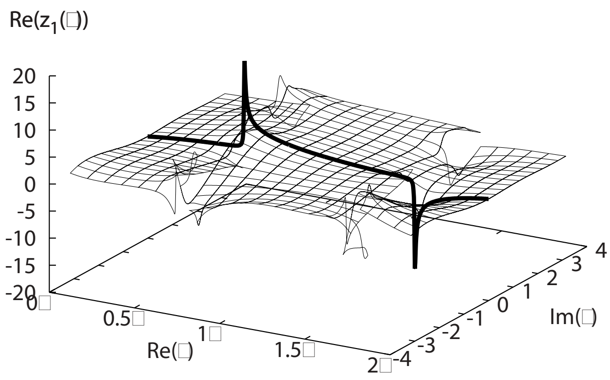

Figure 5: Waveform on complex-time

4. Classifying Periodic Oscillations by Singularity

We classify the periodic oscillations using the movable branch point. Let us consider two paths $\gamma^{(0)}$ and $\gamma^{(1)}$ from $\tau = 0 + 0\mathrm{i}$ to $\tau = T + 0\mathrm{i}$. If there exist no singularities in the closed contour $\gamma^{(0)}$ and $\gamma^{(1)}$, we call that the paths $\gamma^{(0)}$ and $\gamma^{(1)}$ are homotope with respect to one another. Then, the

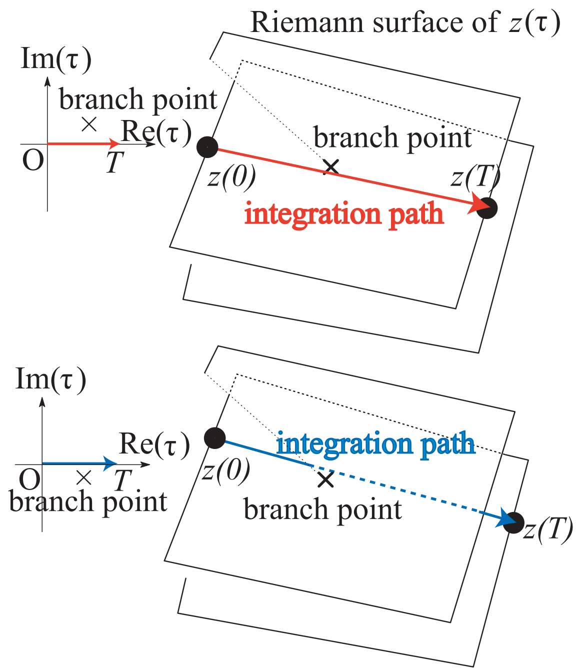

Figure 6: Mechanism of the homotopy path broken by the movable branch point. When we are tracing the general homotopy path, if a movable branch point crosses the real axis on the complex time plane, the $z(T)$ for the real integration paths jumps to the different point on the Riemann surface and the homotopy path is broken.

equivalent relations with respect to the homotope leads to the classification of the paths. For example, the paths $\gamma^{(0)}$ and $\gamma^{(1)}$ in Fig. 4 belong to different classes each other.

Let us classify the solution $z(\tau)$ of Eq.(8) by the concept of the homotope. We assume that the integration path is from $\tau = 0 + 0\mathrm{i}$ to $\tau = T + 0\mathrm{i}$. First, if the initial value $z(0)$ is fixed, the movable branches are fixed and we can classify the solution of Eq.(8) by the concept of the homotope. That is, we classify the solution $z(\tau)$ by the sheet of the Riemann surface of $z(\tau)$. Figure 7 shows the classification of solutions $z^{(0)}(\tau)$, $z^{(1)}(\tau)$ and $z^{(2)}(\tau)$. The class of $z^{(1)}(\tau)$ is different from that of $z^{(0)}(\tau)$ and the class of $z^{(2)}(\tau)$ is equal to that of $z^{(0)}(\tau)$.

Next, we consider the case of different initial values $z^{(0)}(0)$ and $z^{(1)}(0)$. Although the branch points of $z^{(1)}(\tau)$ are different from those of $z^{(0)}(\tau)$, we can identify the sheet by continuous deformation of the solution. That is, introducing a real parameter $\lambda \in [0, 1]$, we connect the two initial values by

$$z(0; \lambda) = \lambda z^{(0)}(0) - (1 - \lambda) z^{(1)}(0). \qquad (10)$$

We vary the initial value $z(0; \lambda)$ from $z^{(1)}(0)$ to $z^{(0)}(0)$ by varying λ from 0 to 1. The waveform $z(\tau; \lambda)$ is calculated on the integration path which is continuously changed as the movable branch points do not cross the path. If

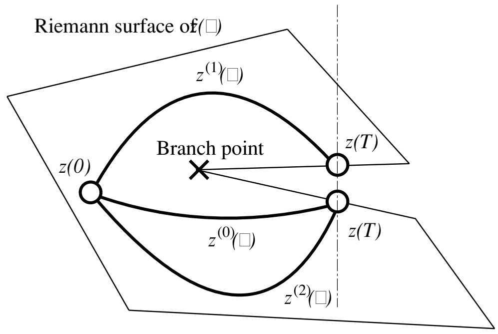

Figure 7: Classification of $z(\tau)$ by the sheet of Riemann surface.

$z(T;1) = z^{(0)}(T) = z(0;1)$ is satisfied at $\lambda = 1$, the two solutions belong to the same class.

Figure 8 shows the test for the classification of S_0 and S_1 in Fig.2. Because $z(2\pi;1) = z(0;1)$ is satisfied at $\lambda = 1$, the two solutions belong to the same class. Figure 9 is the test for S_0 and S_2 and we can confirm that the class of S_2 is different from that of S_0 by $z(2\pi;1) \neq z(0;1)$.

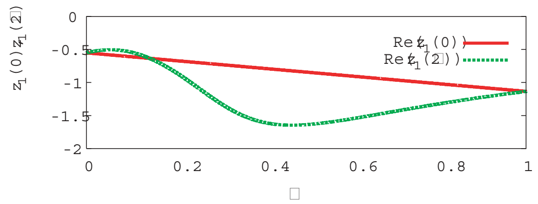

Figure 8: Test for the classification of the solutions S_0 and S_1 in Fig.2. Two solutions belong to the same class

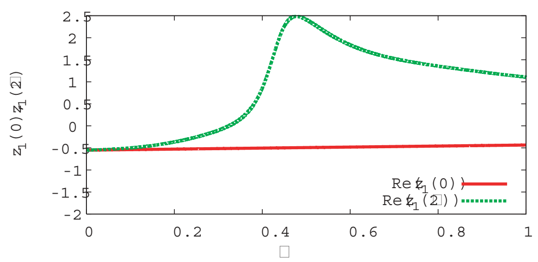

Figure 9: Test for the classification of S_0 and S_2. The class of S_2 is different from that of S_0.

The classification method can be applied to the solutions with different circuit parameters by using the continuous deformation with respect to the circuit parameters. Thus, at least the solutions on connected homotopy path belong to the same class. In the case of Fig.2 the branches which contain S_0, S_1 and S_3 belong to a same class.

5. Limit Search by Classification

Let us consider the real solutions of Eq.(8). We fix the integration path on the real time axis and consider the continuous deformation (10) between two real solutions. Because the real solutions of the circuit equation (8) on the real time axis represent real physical phenomena in the circuit, the real solutions have no singularity on the real time axis. As a result, two real solutions are connected by the continuous deformation (10) and any real solutions belong to a unique class.

Using this property of the real solutions, we propose to extract solutions which have the possibility to reach real solutions before the general homotopy. That is, by testing the class of the periodic solutions obtained by the Newton homotopy, we can discard solutions which have no possibility to reach real solutins.

In order to confirm the efficiency, we apply the test to 1/3-subharmonic solutions of the circuit in Fig.1. At the first step, we fix the parameter (E, ξ, η)=(1.0,0.05,2.8). Second, using the Newton method, we obtain 3 real solutions and 43 complex solutions. By testing the class of the 43 complex solutions, we obtain 12 solutions which belong to the class of real solutions. In this case, the test can decrease the candidates for the general homotopy from 43 to 12. Thus, the test is efficient for decreasing the computational cost of the general homotopy.

6. Conclusion

We extended the time variables of circuit equations to the complex number field and revealed the break of the homotopy path. Then, we proposed to classify the oscillations by the movable branch point on the complex time plane. Last, we proposed to use the classification for extracting the solutions for the general homotopy using the fact that all real solutions belong to a unique class.

References

[1] C. B. Garcia and W. I. Zangwill, "Pathways to Solutions, Fixed Points, and Equilibria", Prentice-Hall, Inc., *Englwood Cliffs*, (1981)

[2] R. Krawczyk, "Newton-algorithmen zur bestimmung von nullstellen mit fehlerschranken," *Computing*, vol. 4, pp. 187–201, 1969.

[3] R.E. Moore, "A test for existence of solutions to nonlinear systems," *SIAM J. Numer. Anal.*, vol. 14, no. 4, pp. 611–615, 1977.

[4] R.E. Moore and S.T. Jones, "Safe starting regions for iterative methods," *SIAM J. Numer. Anal.*, vol. 14, no. 6, pp. 1051–1065, 1977.

[5] T. Hisakado, T.Naruse, K.Okumura, Detection of Periodic Oscillations Using Homotopy with Complex state Variables, Proc. NOLTA, pp. 175-178, (2004)

[6] T.J. Aprile and T.N. Trick, "Steady-State analysis of nonlinear circuits with periodic inputs," *IEEE Proc.*, Vol. 60, No. 1, pp.108-116, 1972.

[7] M.J.ABlowitz and A.S. Fokas, Complex Variables Introduction and Applications Second Edtion, Cambridge University Press, 2003.