2011 International Symposium on Nonlinear Theory and its Applications
NOLTA2011, Kobe, Japan, September 4-7, 2011

Density Functions for Analysis of Positive Invariance and Convergence of Trajectories for Nonlinear Systems with a Piecewise-$\mathbb{C}^2$ Vector Field

Izumi Masubuchi†

†Graduate School of System Informatics, Kobe University
1-1 Rokkodai, Nada, Kobe 657-8501, Japan
Email: msb@harbor.kobe-u.ac.jp

Abstract—This paper considers positive invariance and convergence of trajectories of nonlinear systems via Rantzer's density functions. Unlike many of previous results on density functions, this paper does not assume that the vector field of the system is continuously differentiable on the whole state space but admits vector fields that can be not differentiable in measure-zero subsets. Under mild assumptions, a criterion is proposed to guarantee that a given open set is positively invariant and almost all of the trajectories starting from the set converge to a relevant equilibrium in the set.

1. Introduction

Rantzer's density function [7] has been receiving a considerable amount of attention as a new methodology to stability analysis of nonlinear systems. Similarly to the Lyapunov's direct method, the existence of a density function guarantees a stability, namely, that almost all of the trajectories of the system converge to a relevant equilibrium. A remarkable advantage of density functions lies in computing control inputs with convex formulation [6]. Further theoretical results on density functions have been shown in such as [5, 8, 1, 3]. Most of previous results assume that the vector field of the nonlinear system is continuously differentiable. However, control systems often involve a not-globally-differentiable vector field due to such as saturation and switching. It is then important to relax the assumption on differentiability of the vector field. It can be also noted that in practical systems the vector field is smooth at almost all points other than on certain surfaces of the state space.

Motivated by these observations, in this paper we consider nonlinear systems whose vector field is Lipschitz continuous and piecewise smooth. More precisely, we assume that the vector field is $\mathbb{C}^2$ over the state space other than a closed and measure-zero subset and its second-order derivatives are locally bounded. In terms of piecewise-$\mathbb{C}^2$ density functions, we show a criterion that guarantees that a given open set S is positively invariant and almost all trajectories starting from S converge to a relevant equilibrium in S. This is a generalization of [4] that considers piecewise-$\mathbb{C}^2$ vector fields with a global setting (i.e., S is the whole state) and also that of [3] to Lipschitz continuous and piecewise-$\mathbb{C}^2$ vector fields.

Notation. Let $\mathbf{R}$ and $\mathbf{Z}$ denote real numbers and integers, respectively. For a subset $S \subset \mathbf{R}^n$, $\overline{S}$, S^{o}, ∂S and S^{c} stand for the closure, the interior, the boundary and the complement of S, respectively. For vectors and matrices, let $\|\cdot\|$ denote the Euclid and maximal-singular-value norms, respectively. For $x \in \mathbf{R}^n$ and $r > 0$, denote by $B(r;x)$ and $\overline{B}(r;x)$ the open and closed ball with center x and radius r, respectively. For a subset X of $\mathbf{R}^n$, $\mathbb{B}(r;X) = \bigcup\{B(r;x) : x \in X\}$ and $\overline{\mathbb{B}}(r;X) = \overline{\mathbb{B}(r;X)}$. Note that $\mathbb{B}(r;X)$ is open. For subsets $X, Y \subset \mathbf{R}^n$, $\mathrm{dist}(X,Y) = \inf\{\|x-y\| : x \in X, y \in Y\}$. Denote by $\mathbb{C}^k(X,Y)$ the set of k-times continuously differentiable functions from X to Y. Denote simply $\mathbb{C}(X,Y) = \mathbb{C}^0(X,Y)$, which is the set of continuous functions. A proposition $P(x)$ stated on $x \in \mathbf{R}^n$ holds for almost every (a. e.) $x \in S$ for a subset $S \subset \mathbf{R}^n$ if the Lebesgue measure of the set $\{x \in \mathbf{R}^n : P(x)$ is not true$\}$ is zero.

2. Main result

Consider the following nonlinear system:

$$\dot{x} = f(x), \quad x \in \mathbf{R}^n. \tag{1}$$

We assume in (1) that the origin is an equilibrium ($f(0) = 0$) and $f(x)$ is Lipschitz continuous and piecewise-$\mathbb{C}^2$ on $\mathbf{R}^n$, where we define piecewise differentiability below:

Definition 1 *A function $f(x)$ is* piecewise-$\mathbb{C}^k$ *on an open set $S \subset \mathbf{R}^n$ if there exists a closed set $E_f \subset \mathbf{R}^n$ with $\mu(E_f) = 0$ such that $f(x) \in \mathbb{C}^k(S \setminus E_f, \mathbf{R}^n)$ and for every compact subset $A \subset S$ all the relevant derivatives are bounded as:*

$$\sup_{x \in A \setminus E_f} \left\| \frac{\partial^i f}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} \right\| < \infty \tag{2}$$

for $j_1, \ldots, j_n \geq 0$ such that $i := j_1 + \cdots + j_n \in \{0, \ldots, k\}$.

The Lipschitz continuity guarantees that there exists a unique solution $x(t)$ to (1) satisfying $x(0) = x_0$ for every initial state $x_0 \in \mathbf{R}^n$ locally around $t = 0$ [2]. Denote by $\varphi(t;x_0)$ the solution $x(t)$ satisfying $x(0) = x_0$.

Definition 2 *A set $S \subset \mathbf{R}^n$ is said to be* positively invariant *if $\varphi(t;x) \in S$ holds for all $x \in S$, $t \geq 0$.*

Definition 3 *The system (1), which is supposed to have an equilibrium at the origin, is* almost everywhere stable *on a set S if for almost every initial value $x \in S$ the trajectory $\varphi(t;x)$ exists for all $t \in [0,\infty)$ and converges to the origin as $t \to \infty$.*

The following criterion was shown in the original work of Rantzer [7] to guarantee almost-everywhere stability on $\mathbf{R}^n$ of nonlinear systems (1) with a $\mathbb{C}^1$ vector field in terms of *density functions*.

Lemma 1 *Suppose that $f \in \mathbb{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^n)$ and $\|f\|/\|x\|$ is bounded near the origin. Assume that there exists a nonnegative function $\rho \in \mathbb{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ for which $f\rho/\|x\|$ is integrable on $\{x \in \mathbf{R}^n : \|x\| > 1\}$ and $[\nabla \cdot (f\rho)](x) > 0$ holds for almost all $x \in \mathbf{R}^n$. Then the system (1) is almost everywhere stable.*

For a measurable set $A \subset \mathbf{R}^n$, denote

$$\mu_\rho(A) = \int_A \rho(x)\mathrm{d}x. \tag{3}$$

In Lemma 1, the positivity of $\nabla \cdot (f\rho)$ implies the monotonous increase of $\mu_\rho(\varphi(t;A))$ with respect to t, as a consequence of the following lemma [7].

Lemma 2 *Let $M \subset \mathbf{R}^n$ be open, $A \subset M$ be measurable and $f \in \mathbb{C}^1(M, \mathbf{R}^n)$, $\rho \in \mathbb{C}^1(M, \mathbf{R})$. Suppose that $\varphi(t;A)$ is included in M for all $t \in [0,T]$. Then*

$$\int_{\varphi(T;A)} \rho(x)\mathrm{d}x - \int_A \rho(x)\mathrm{d}x = \int_0^T \int_{\varphi(t;A)} [\nabla \cdot (f\rho)](x)\mathrm{d}x\mathrm{d}t. \tag{4}$$

This equality, which plays one of the key roles in [7], is not available because f is not assumed to be continuously differentiable in (1). However, an alternative inequality is proved in [4], which will turn out to suffice to prove the main result of the paper.

Lemma 3 *Let $U \subset \mathbf{R}^n$ be a closed set and let $f(x) : x \in U \to \mathbf{R}^n$ and $\rho(x) : x \in U \to \mathbf{R}$ be bounded functions on U that are Lipschitz continuous and piecewise-$\mathbb{C}^2$ on U°. Let E be a closed set such that $\mu(E) = 0$ and f and ρ are twice continuously differentiable on $U^\circ \setminus E$. Let $A \subset U$ be a Borel set of $\mathbf{R}^n$ that has a finite rank*[1] *and assume $\varphi(t;A) \in U$ for all $t \in [0,T]$, where $T > 0$. Suppose that $\rho(x) \geq 0$ for all $x \in A$ and $[\nabla \cdot (f\rho)](x) \geq 0$ for a. e. $x \in A \setminus E$. Then*

$$\int_{\varphi(T;A)} \rho(x)\mathrm{d}x - \int_A \rho(x)\mathrm{d}x \geq \int_0^T \int_{\varphi(\tau;A)\setminus E} [\nabla \cdot (f\rho)](x)\mathrm{d}x\mathrm{d}\tau. \tag{5}$$

and hence $\mu_\rho(\varphi(t;A))$ is monotonously increasing for $t \in [0,T]$.

The inequality (5) has been used to prove the main result (Theorem 1) of [4] that generalizes Lemma 1 to nonlinear systems (1) whose vector field is Lipschitz continuous and piecewise-$\mathbb{C}^2$ in order to handle almost-everywhere stability on the whole state space. Below we further extend Theorem 1 of [4] and provide a condition under which an open set S is positively invariant for the system (1) and the nonlinear system (1) is almost everywhere stable on S.

Theorem 1 *Consider the nonlinear system (1) whose vector field f is Lipschitz continuous and piecewise-$\mathbb{C}^2$ on $\mathbf{R}^n$. Let $S \subset \mathbf{R}^n$ be open set with $\mu(\partial S) = 0$ and $0 \in S$. Suppose that there exists a function $\rho(x) : x \in S \setminus \{0\} \to \mathbf{R}$ such that ρ is Lipschitz continuous and piecewise-$\mathbb{C}^2$ on $S \setminus \{0\}$ and satisfies*

$$\rho(x) > 0 \quad \forall x \in S \setminus \{0\}, \tag{6}$$

$$[\nabla \cdot (f\rho)](x) > 0, \quad \textit{a. e. } x \in S \setminus E, \tag{7}$$

$$\lim_{x \to \partial S} \rho(x) = 0, \tag{8}$$

where $E = E_f \cup E_\rho \cup \partial S \cup \{0\}$ with E_ρ being a closed set such that $\rho \in \mathbb{C}^2(S \setminus E_\rho)$ and $\mu(E_\rho) = 0$. Then (1°) S is positively invariant. Moreover, (2°) if

$$\int_{S \setminus N_o} (1 + \|f(x)\|^2)\rho(x)\mathrm{d}x < \infty \tag{9}$$

for some bounded neighborhood of the origin N_o, then the system (1), which has an equilibrium at the origin, is almost everywhere stable on S.

This is also a generalization of [3], which considers positive invariance via density functions with $\mathbb{C}^1$ vector field, to nonlinear systems with a Lipschitz continuous and piecewise-$\mathbb{C}^2$ vector field. See also [8].

Note that the integrability condition (9) always holds if S is bounded. Obviously, $(1 + \|f\|^2)\rho$ is integrable also on $S \setminus N$ for *any* neighborhood N of the origin if (9) is true.

Example 1 Let us consider a piecewise linear system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + 4x_2 - 2|x_1| \\ -6x_1 - 3x_2 + |x_2| \end{bmatrix}, \tag{10}$$

which is Lipschitz continuous and smooth on E_f^{c}, where E_f is the union of x_1- and x_2-axes. It is easy to verify that

$$\rho(x) = \max\left\{ \frac{1}{(x_1^2 + 0.0707x_1x_2 + 0.6934x_2^2)^5} - c, 0 \right\}$$

satisfies the assumptions of the theorem for any $c > 0$, proving positive invariance of $S = \{x \in \mathbf{R}^2 : x_1^2 + 0.0707x_1x_2 + 0.6934x_2^2 < c^{-5}\}$.

[1] A Borel set with a finite rank is a set represented as finitely repeated infinite union or intersection of open or closed sets, i.e., those represented as $\Pi_{i_1}^1 \cdots \Pi_{i_k}^k A_{i_1,\ldots,i_k}$ where k is a (finite) positive integer, $\Pi_{i_j}^j$ stands for $\bigcup_{i_j=1}^\infty$ or $\bigcup_{i_j=1}^\infty$ and sets $A_{i_1,\ldots,i_k}$ are open or closed sets.

3. Proof of Theorem 1

First, extend $\rho(x)$ for $x \in S^{\mathrm{c}}$ by setting $\rho(x) = 0$, $x \in S^{\mathrm{c}}$. Then from (8) $\rho(x)$ is Lipschitz continuous and piecewise-$\mathbb{C}^2$ on $\mathbf{R}^n \setminus \{0\}$ and the non-differentiable points of ρ are included in $E_\rho \cup \partial S$, whose Lebesgue measure is zero. Obviously $[\nabla \cdot (f\rho)](x) = 0$ for all $x \in (\overline{S})^{\mathrm{c}}$.

Next, define

$$\begin{cases} \bar{f}(x) = f(x)/(1 + \|f(x)\|^2), \\ \bar{\rho}(x) = (1 + \|f(x)\|^2)\rho(x). \end{cases} \tag{11}$$

Then $\bar{f}$ and $\bar{\rho}$ satisfy all the assumptions so far assumed for f and ρ, respectively, instead of the integrability condition (9) but

$$\int_{\mathbf{R}^n \setminus N_o} \bar{\rho}(x)\mathrm{d}x < \infty \tag{12}$$

Moreover, if we denote by $\bar{\varphi}(\tau; x_0)$ the solution of $\mathrm{d}y/\mathrm{d}\tau = \bar{f}(y)$ with $y(0) = x_0$, the solution $\bar{\varphi}(\tau; x)$ is defined for all $\tau \in \mathbf{R}$ since $\bar{f}$ is Lipschitz continuous on $\mathbf{R}^n \setminus \{0\}$. It is easy to see that $\bar{\varphi}(\tau; x_0) = \varphi(t; x_0)$ if $\tau = \int_0^t (1 + \|f(\varphi(s; x_0))\|^2)\mathrm{d}s$. Hence the system (1) and $\mathrm{d}y/\mathrm{d}\tau = \bar{f}(y)$ share the same set of integral curves and the convergence of one implies that of the other. Thus we can prove Theorem 1 for $\bar{f}$ and $\bar{\rho}$ with the integrability assumption (12). To simplify the notation below we denote (f, ρ) instead of $(\bar{f}, \bar{\rho})$.

(1°) Since $\varphi(t; x)$ is continuous in t and S is open, if S is *not* positively invariant, there exists $x \in S$ such that $x_1 := \varphi(t_1; x) \in \partial S$ for some $t_1 > 0$. Let N be a neighborhood of the origin such that $\overline{N} \subset S$ and $x \notin \overline{N}$. Since $S \setminus \overline{N}$ is open, there exists a neighborhood A of x such that $A \subset S \setminus \overline{N}$. Let $A_1 := \varphi(t_1; A)$ and $A_{1o} := A_1 \cap (\overline{S})^{\mathrm{c}}$, which are open because $\varphi(\cdot; \cdot)$ is a homeomorphism with respect to the first variable and $(\overline{S})^{\mathrm{c}}$ is open. Since A_1 is a neighborhood of $x_1 \in \partial S$, A_{1o} is not empty. Moreover, $A_o = \varphi(-t_1; A_{1o})$ is also nonempty and open. From Lemma 3, we have

$$\mu_\rho(A_o) \le \mu_\rho(\varphi(t_1; A_o)) = \mu_\rho(A_{1o}) = 0,$$

which means that for the nonempty open set $A_o \subset A \subset S \setminus \overline{N}$ the integral $\int_{A_o} \rho(x)\mathrm{d}x = 0$. This implies that $\rho(x) = 0$ for almost all x in the nonempty open set A_o, contradicting (6). Thus we conclude that S is positively invariant.

(2°) First, we refer to a lemma shown in [4], which is a version of Theorem 2 in [7].

Lemma 4 *Consider the measure space $(\mathbf{R}^n, \mathcal{B}, \mu)$, where $\mathcal{B}$ stands for the Borel family and μ is a measure. Let P be an open set with $\mu(P) < \infty$ and $\mathcal{T}$: $\mathbf{R}^n \to \mathbf{R}^n$ be a homeomorphism and assume that $\mathcal{T}$ is measurable. Suppose that $\mu(\mathcal{T}A) \ge \mu(A)$ holds for all $G_{\sigma\delta}$- and $F_{\delta\sigma}$-sets A with $\mu(A) < \infty$. Define Z as the set of elements $x \in P$ such that $\mathcal{T}^k(x) \in P$ for infinitely many integers $k \ge 0$. Then $\mu(\mathcal{T}Z) = \mu(Z)$.*

To apply Lemma 4, fix $r > 0$ and define

$$\begin{aligned} X &= \mathbf{R}^n, \\ P &= \{x \in S : \|x\| > r\}, \\ Z &= \{x \in P : \|\varphi(k; x)\| > r, \\ &\qquad \text{for infinitely many integers } k \ge 0\}, \\ \mathcal{T}(x) &= \varphi(1; x), \\ \mu(A) &= \int_A \rho(x)\mathrm{d}x. \end{aligned}$$

Lemma 3 tells that $\mu(\mathcal{T}A) \ge \mu(A)$ holds for all Borel sets A with finite rank if $\mu(A) < \infty$, which is equivalent to $\mathrm{dist}(A, \{0\}) > 0$. Therefore, from Lemma 4 we have $\mu(\mathcal{T}Z) = \mu(Z)$, i.e.,

$$\begin{aligned} 0 &= \int_{\varphi(1;Z)} \rho(x)\mathrm{d}x - \int_Z \rho(x)\mathrm{d}x \\ &\ge \int_0^1 \int_{\varphi(\tau;Z)\setminus E} [\nabla \cdot (f\rho)](x)\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{13}$$

Since $[\nabla \cdot (f\rho)](x) > 0$ for almost all $x \in S \setminus E$, (13) implies $\mu(S \cap \varphi(\tau; Z) \setminus E) = 0$ for almost all $\tau \in [0, 1]$ and hence $\mu(S \cap \varphi(\tau; Z)) = 0$ for a. a. $\tau \in [0, 1]$ since $\mu(E) = 0$. Moreover, we can also deduce $\mu(S \cap Z) = 0$ (See the appendix). Hence for almost all $x \in P$ there exists an integer k such that $\varphi(j; x) \in P$ i.e., $\|\varphi(j; x)\| \le r$ for all $j \ge k$. The rest of the proof follows similarly to that of Theorem 2 in [7]; we also have $\lim_{j\in\mathbf{Z}, j\to\infty} \|\varphi(j; x)\| = 0$ since the above discussion is valid for any $r > 0$. Moreover $\lim_{t\in\mathbf{R}, t\to\infty} \|\varphi(t; x)\| = 0$ holds because $\|f(x)\|/\|x\|$ is bounded near 0 from Lipschitz continuity. Thus the proof of Theorem 1 is completed. ■

4. Conclusion

We considered nonlinear systems whose vector filed is Lipschitz continuous and piecewise-$\mathbb{C}^2$ and for such systems we presented conditions in terms of density functions under which a set is positively invariant and almost all of trajectories starting from there converge to the origin. This result, which is an generalization of [3, 4], allows us to handle a larger class of nonlinear systems that do not necessarily have a $\mathbb{C}^1$ vector field.

References

[1] D. Angeli, "An almost global notion of input-to-state stability," *IEEE Transactions on Automatic Control*, Vol. 49, No. 6, pp. 866-874 (2004)

[2] H. K. Khalil, *Nonlinear systems, third edition*, Prentice Hall (2002)

[3] I. Masubuchi, "Analysis of positive invariance and almost regional attraction via density functions with converse results," *IEEE Transactions on Automatic Control*, Vol. 52, No. 7, pp. 1329-1333 (2007)

[4] I. Masubuchi, "Almost-everywhere stability of systems with a piecewise-$\mathbb{C}^2$ Lipschitz continuous vector field," *Proceedings of the 18th IFAC World Congress*, ThA26.11 (2011) *to be presented*

[5] P. Monzon, "On necessary conditions for almost global stability," *IEEE Transactions on Automatic Control*, Vol. 48, No. 4, pp.631-634 (2003)

[6] S. Prajna, A. Parrilo and A. Rantzer, "Nonlinear control synthesis by convex optimization," *IEEE Transactions on Automatic Control*, Vol. 49, No. 2, pp.310-314 (2004)

[7] A. Rantzer, "A dual to Lyapunov's stability theorem," *Systems & Control Letters*, Vol. 42, pp. 161-168 (2001)

[8] A. Rantzer and S. Prajna, "On analysis and synthesis of safe control laws," *Proceedings of the 42nd Annual Allerton Conference on Communication, Control and Computing* (2004)

[9] O. Takenouchi: *Lebesgue Integral*, Baifukan (1980) in Japanese

Appendix

A. Preliminaries

Here we make several preparations for the proof of $\mu(S\cap Z)=0$, which is shown in Section B.

Definition 4 *For a set $A\subset\mathbf{R}^n$ and positive number r, define $\underline{\mathbb{B}}(r;A):=\mathbb{B}(r;A^{\rm c})^{\rm c}$.*

Obviously $\underline{\mathbb{B}}(r;A)\subset A$ for any $r\geq 0$ since $\mathbb{B}(r;A^{\rm c})\supset A^{\rm c}$. Similarly $r_1<r_2$ implies $\underline{\mathbb{B}}(r_1;A)\supset\underline{\mathbb{B}}(r_2;A)$. Since $\mathbb{B}(r;A^{\rm c})$ is always open [9], $\underline{\mathbb{B}}(r;A)$ is closed.

Lemma 5 *Suppose that $\|f(x)\|\leq f_{\max}$ for all x and $h>f_{\max}$. Then $\underline{\mathbb{B}}(h|t|;A)\subset\varphi(t;A)$.*

Proof. Assume $x\in A^{\rm c}$. Since $\|f(x)\|\leq f_{\max}$, it holds that $\|\varphi(t;x)-x\|\leq f_{\max}|t|<h|t|$, which implies $\varphi(t;x)\in B(h|t|;x)\subset\mathbb{B}(h|t|;A^{\rm c})$. This means $\varphi(t;A)^{\rm c}=\varphi(t;A^{\rm c})\subset\mathbb{B}(h|t|;A^{\rm c})$ and hence $\varphi(t;A)\supset\mathbb{B}(h|t|;A^{\rm c})^{\rm c}=\underline{\mathbb{B}}(h|t|;A)$. ∎

Lemma 6 *Suppose that a sequence of monotonously decreasing positive numbers a_i satisfies $\lim_{i\to\infty}a_i=0$ and that A is an open set of $\mathbf{R}^n$. Then*

$$\bigcup_{i=1}^{\infty}\underline{\mathbb{B}}(a_i;A)=A \tag{14}$$

Proof. First, let us invoke the following lemma from standard results on sets [9]:

Lemma 7 *Suppose that $a_i>0$ is monotonously decreasing and satisfy $\lim_{i\to\infty}a_i=0$ and $S\subset\mathbf{R}^n$. Then $\bigcap_{i=1}^{\infty}\mathbb{B}(a_i;A)=\overline{A}$.*

Proof of Lemma 6. From Lemma 7 and the assumption that A is open, $\bigcap_{i=1}^{\infty}\mathbb{B}(a_i;A^{\rm c})=\overline{(A^{\rm c})}=A^{\rm c}$. Therefore $A=\bigcup_{i=1}^{\infty}\mathbb{B}(a_i;A^{\rm c})^{\rm c}=\bigcup_{i=1}^{\infty}\underline{\mathbb{B}}(a_i;A)$. Thus (14) is proved. ∎

B. Proof of $\mu(S\cap Z)=0$

We have seen $\mu(S\cap\varphi(\tau;Z))=0$ for almost all $\tau\in[0,1]$. Then there exists a sequence of real numbers $a_1,a_2,\dots$ such that

$$a_k\in(0,1/k),\quad \mu(S\cap\varphi(a_k;Z))=0;$$

otherwise $\mu(S\cap\varphi(\tau;Z))>0$ for all $\tau\in[0,1/k)$ for some k, which is a contradiction. Without loss of generality, we can assume that a_k is monotonously decreasing. Let

$$Z_k=\underline{\mathbb{B}}(a_k;S)\cap Z\cap\overline{\mathbb{B}}(1/k;E)^{\rm c},\quad k=1,2,\dots$$

Then from Lemmas 6 and 7,

$$\begin{aligned}\bigcup_{k=1}^{\infty}Z_k &= \bigcup_{k=1}^{\infty}\left\{\underline{\mathbb{B}}(a_k;S)\cap Z\cap\overline{\mathbb{B}}(1/k;E)^{\rm c}\right\}\\ &= \left\{\bigcup_{k=1}^{\infty}\underline{\mathbb{B}}(a_k;S)\right\}\cap Z\cap\left\{\bigcup_{k=1}^{\infty}\overline{\mathbb{B}}(1/k;E)^{\rm c}\right\}\\ &= S\cap Z\cap E^{\rm c}.\end{aligned}$$

Since $\|f(x)\|\leq 1/2$ for all $x\in\mathbf{R}^n$ from (11), $\|x-\varphi(\tau;x)\|\leq|\tau|/2$. Therefore, if $x\in Z_k\subset\overline{\mathbb{B}}(1/k;E)^{\rm c}$, it holds that $\varphi(\tau;x)\notin E$ for all $\tau\in[-1/k,1/k]$, i.e., $\varphi(\tau;Z_k)\cap E=\emptyset$ for all $\tau\in[-1/k,1/k]$. This allows us to exploit differentiability of $\varphi(t;x)$ with respect to x: since $a_k\in(0,1/k)$, Lemma 5 yields

$$\begin{aligned}0 &= \int_{S\cap\varphi(a_k;Z)}{\rm d}x \;\geq\; \int_{S\cap\varphi(a_k;Z_k)}{\rm d}x\\ &= \int_{\varphi(-a_k;S)\cap Z_k}\det\left(\frac{\partial\varphi(a_k;x)}{\partial x}\right){\rm d}x\\ &\geq \int_{\underline{\mathbb{B}}(a_k;S)\cap Z_k}\det\left(\frac{\partial\varphi(a_k;x)}{\partial x}\right){\rm d}x\\ &= \int_{Z_k}\det\left(\frac{\partial\varphi(a_k;x)}{\partial x}\right){\rm d}x\geq 0,\end{aligned}$$

which implies $\mu(Z_k)=0$. Lastly

$$\begin{aligned}\mu(S\cap Z) &= \mu(S\cap Z\cap E^{\rm c})+\mu(S\cap Z\cap E)\\ &\leq \sum_{i=1}^{\infty}\mu(Z_k)=0.\end{aligned}$$