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Analyses of Coupled Hindmarsh-Rose Type Bursting Oscillators

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Abstract— We proposed the Inverse function Delayed model(ID model) as one of neuron models [1]. In addition, we have propose Burst firing ID model [2] of Hodgkin-Huxley(H-H) type model that has the burst oscillating characteristics with three variables, because we consider that the burst dynamics has prospects of capabilities of effective tool for information processing consequently. The burst ID model is explained with a new concept that the neuron dynamics is expressed as a motion of a quasi particle in a potential with active areas, and then we are able to apply the concept to others. Through the technique, we are able to foresee the landscape of solutions with the curvature of the potential and design the wave forms if we place the active area on the potential properly [3]. In this paper, we apply this concept to coupling systems and the concept is effective in the interconnected systems too, and analyze the dynamics of the coupled models with the burst firing characteristics .

1. Introduction

We proposed the Inverse function Delayed model(ID model) as one of neuron models [1]. In addition, we have proposed Burst firing ID model of Hodgkin-Huxley(H-H) type model that has the burst oscillation characteristics with three variables, because we consider that the burst dynamics has prospects of capabilities of effective tool for information processing consequently. The burst ID model is explained with a new concept that the neuron dynamics is expressed as a motion of a quasi particle in a potential with active areas, and then we are able to apply the concept to others. We are able to obtain the landscape of the solution with the curvature of the potential. If the curvature is positive as a whole, the wave forms of the solutions are oscillations or resting state without divergence. On the contrast, when the curvature is negative, we can observe the divergence solution.

This paper is organized as follows. We explain our concept of potential with the active area in the next section. We analyze the dynamics with van der Pol model as one of the simplest nonlinear oscillator, and our concept is effective for analyses of coupled system in Sec.3. Last section, we use Hindmarsh-Rose type model for the analyses of the dynamics of coupled bursting oscillators.

2. Potential with Active Area

Neuron models are typically expressed in the form of multidimensional nonlinear differential equation. Several models can be transformed into higher differential equation with one variable.

$$\frac{d^n x}{dt^n} + b_{n-1}(x, \frac{dx}{dt}, \cdots, \frac{d^{n-1}}{dt^{n-1}})\frac{dx}{dt} + \cdots + b_1(x, \frac{dx}{dt})\frac{dx}{dt} = F(x,\theta) = -\frac{\partial U(x,\theta)}{\partial x} \qquad (1)$$

$U(x)$ is a kind of the potential function, and has some equilibria x_0 that depend on the external input θ. We have obtained the characteristic equation and analyze the stability at the neighborhood of equilibria determined in accordance with Hurwitz's theorem, and the condition of x_0 are stable(n=4), if

$$b_0(x_0) = \frac{d^2U(x_0)}{dx^2} > 0, \qquad (2)$$

$$b_i(x_0) > 0, (i = 1, 2, 3) \qquad (3)$$

$$B_1(x_0) = b_2(x_0, 0, 0)b_1(x_0, 0) - b_0(x_0)b_3(x_0, 0, 0, 0), > 0 \qquad (4)$$

$$B_2(x_0) = B_1(x_0)b_1(x_0, 0) - b_3(x_0, 0, 0)^2 b_0(x_0) > 0. \qquad (5)$$

Equation (2) means that the curvature b_0 of the potential $U(x,\theta)$ is positive at x_0. An equilibrium points is unstable where $b_1(x) < 0$ or $b_2(x) < 0$ or $b_3(x) < 0$ or $B_1(x) < 0$ or $B_2(x) < 0$ is satisfied even if the global curvature of the potential is positive. In other words, if there is no equilibrium point satisfy the requirement of Hurwitz's theorem and the global curvature is positive, it causes various oscillations, for example, periodic limit cycle, burst firing and so on. We identify the section, where coefficients and these minors are negative, as active area. The section of x where $b_1(x) < 0$ is satisfied represents a negative damping area, we call this area b_1 active area.

3. van der Pol Model

3.1. Basic Equation

we explain our concept of potential and active areas with stand-alone van der Pol model. Generally this model is expressed as following equation

$$\frac{d^2x}{dt^2} - q(1-x^2)\frac{dx}{dt} = -x. \tag{6}$$

In this paper, we consider the following model equation of van der Pol type, because we can set the active area arbitrarily considering the external input and time-delay.

$$\tau\frac{d^2x}{dt^2} + \epsilon\{(x-\alpha)^2 - \beta\}\frac{dx}{dt} = Wx + Z = -\frac{\partial U(x,\theta)}{\partial x}. \tag{7}$$

We express in the form of differential equation with two variable,

$$\tau\frac{dx}{dt} = u - \epsilon\{\frac{1}{3}x^3 - \alpha x^2 + (\alpha^2 - \beta)x\}$$
$$\frac{du}{dt} = Wx + \theta. \tag{8}$$

Where x, u, W, θ, and τ are the output of the unit, the internal state, the self-connection, the external input, the time constant respectively, α,β and ϵ are control parameters. $U(x,\theta)$ is a kind of the potential function, described as

$$U(x) = -\frac{1}{\tau}(\frac{1}{2}Wx^2 + \theta x). \tag{9}$$

Potential function has one equilibrium point x_0 and the curvature of potential $b_0 = \frac{\partial^2 U(x,\theta)}{\partial x^2} = -W$. Therefore the potential function became a convex function if $W > 0$ and we obtain a divergence solution. In contrast, the potential is concave function if $W < 0$. The equilibrium point is located internally in active area, the output is continuous oscillation in this case. This system expressed by Eq.(7) has one active area b_1 active area.

$$b_1(x) = \frac{\epsilon}{\tau}\{(x-\alpha)^2 - \beta\}. \tag{10}$$

The b_1 active area is the section of x that where $(x-\alpha)^2-\beta < 0$ satisfied, where $\beta > 0$, α and β denote parameters that define the middle of b_1 active area and its width.

3.2. Interconnected Models

We discuss the interconnected system through technique noted above, and we interconnect two units described by Eq.(7), however the interconnection and self-connection are equal($\omega_{ij} = \omega_{ji} = \omega, W_x = W_y = W$),

$$\tau_x\frac{d^2x}{dt^2} + \epsilon_x\{(x-\alpha_x)^2 - \beta_x\}\frac{dx}{dt} = Wx + \omega y + \theta_x, \tag{11}$$

$$\tau_y\frac{d^2y}{dt^2} + \epsilon_y\{(y-\alpha_y)^2 - \beta_y\}\frac{dy}{dt} = Wy + \omega x + \theta_y. \tag{12}$$

Equation (11) and (12) can be transformed into the differential equation with one variable x

$$\frac{d^4x}{dt^4} + b_3(x)\frac{d^3x}{dt^3} + b_2(x,\dot{x})\frac{d^2x}{dt^2} + b_1(x,\dot{x})\frac{dx}{dt} = -\frac{\partial U(x)}{\partial x}, \tag{13}$$

where

$$b_1(x) = -\frac{W}{\tau_y}\eta(x) - \frac{W}{\tau_x}\zeta(x), \tag{14}$$

$$b_2(x) = -\frac{W}{\tau_x} - \frac{W}{\tau_y} + \eta(x)\zeta(x), \tag{15}$$

$$b_3(x) = \eta(x) + \zeta(x), \tag{16}$$

$$B_1(x) = b_1(x)b_2(x) - b_3(x)b_0(x), \tag{17}$$

$$B_2(x) = B_1(x)b_1(x) - b_3(x)^2 b_0(x), \tag{18}$$

where $\eta(x) = \frac{\epsilon_x}{\tau_x}\{(x-\alpha_x)^2-\beta_x\}, \zeta(x) = \frac{\epsilon_y}{\tau_y}\{(\frac{Wx+\theta_x}{\omega}+\alpha_y)^2-\beta_y\}$. We can also obtain the potential function

$$U(x) = \frac{1}{\tau_x\tau_y}\{\frac{1}{2}(W^2-\omega^2)x^2 + (W\theta_x - \omega\theta_y)x\}. \tag{19}$$

This potential has one equilibrium point as stand-alone model, and the equilibrium point x_0 is

$$x_0 = -\frac{W\theta_x - \omega\theta_y}{W^2 - \omega^2}, \tag{20}$$

and it also depends on the external inputs and the connection strength. The curvature of the potential b_0 is

$$b_0(x) = \frac{1}{\tau_x\tau_y}(W^2 - \omega^2), \tag{21}$$

and hence we obtain a divergence solution if $| W | < | \omega |$. When the equilibrium point is located internally in some of the five active areas described by Eq.(14)~Eq.(18) with $| W | > | \omega |$, we can obtain continuous oscillations.

3.3. Fast-Slow Dynamics

We simulate the coupled system where $\beta_y = 1.0, \epsilon_x = 3.0, \epsilon_y = 1.0, \theta_y = 0.0, \tau_x = 1.0, \tau_y = 100$, and observe the output x changing the parameters β_x and θ. Figure.1 shows the output and the position of active areas for the equilibrium point expressed by Eq.(20) of the quadratic potential. The line L1 and L2 denote parameter sets when the equilibrium point of the potential is on the edge of active areas. It is within B_1 active area all over the region in Fig.1. It is within b_1 active area and b_2 active area above L1 and L2. We are able to observe various oscillations near by L2. For example, spike pulse(SP), oscillation with two different frequencies (SFO) and bursting oscillation(FB). If we change the parameter θ_x under an identical condition about parameter β_x. In region (a), we observe the slow oscillation as a solution, we increase the external input θ_x and cross the Line 1, then we are able to get the spiking pulse in very narrow region of diagram. When the more the equilibrium point of the potential approaches b_3 active area with increasing the external input, the number of spikes per burst increases generated in a time between resting states. Figure (2) shows various wave forms. In other words, if the potential is single well form and there are active areas with two different frequency, we are able to obtain the spiking or bursting oscillation to set the active area with row frequency on the equilibrium point and the active area of fast oscillation near the equilibrium point.

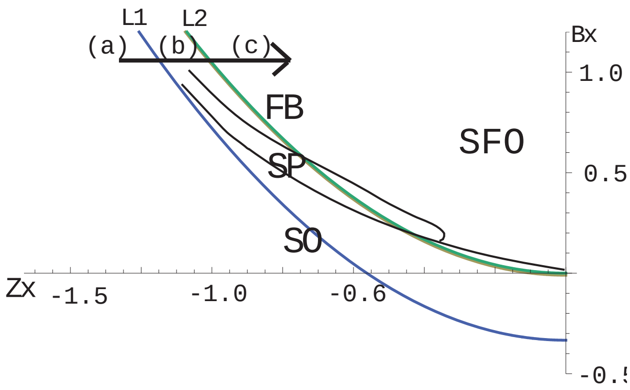

Figure 1: Parameter diagram of coupled van der Pol oscillators. SP,SO,SFO,FB denote the spike pulse, the slow oscillation, the oscillation with two different frequencies and the burst firing with undershoot.

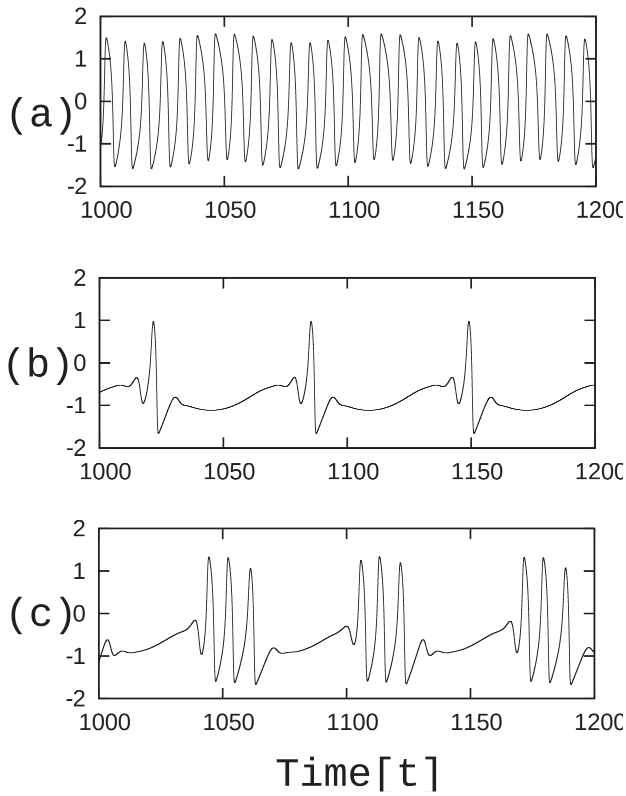

Figure 2: Time series of the output $x(t)$ for coupled van der Pol model with β_x=0.55. $\theta_x = -1.00$ (a), $\theta_x = -0.80$ (b),$\theta_x = -0.60$ (c).

4. Coupled Bursting Oscillators

4.1. Basic Equations for Hindmarsh-Rose Type Model

We are interested in coupled bursting oscillators, considering that our concept is very effective for the analyses of these systems. Therefore, we choice Hindmarsh-Rose Type[4] model as a bursting oscillator. We calculate the following equations, because we are able to set the active area optionally, interconnection between the units and time constant of units.

$$\tau\frac{dx}{dt} = u - z - g(x) + I, \tag{22}$$

$$\frac{du}{dt} = -u - W(x) + \theta, \tag{23}$$

$$\frac{dz}{dt} = r\{Z(x) - z\}, \tag{24}$$

$$g(x) = \epsilon\{\frac{1}{3}x^3 - \alpha x^2 + (\alpha^2 - \beta)x\} \tag{25}$$

$$W(x) = dx^2 \tag{26}$$

$$z(x) = s(x - z_0) \tag{27}$$

where x denotes the output, $\alpha, \beta, \epsilon, d, r, s$ and z_0 denotes the control parameters, I denotes an external stimulate, θ is the bias and τ is the time constant. $g(x), W(x)$ and $Z(x)$ are defined according to the second section, we transform Eq.(22)〜(24) into the following single equation.

$$\frac{d^3x}{dt^3} + b_2(x)\frac{d^2x}{dt^2} + b_1(x,\dot{x})\frac{dx}{dt} = -\frac{r}{\tau}(g(x) + z(x) + W(x) - I - \theta), \tag{28}$$

where $b_2(x)$ and $b_1(x)$ functions are expressed as

$$b_2(x) = \frac{1}{\tau}\epsilon\{(x-\alpha)^2 - \beta\} + r + 1, \tag{29}$$

$$b_1(x) = \frac{1}{\tau}\{(r+1)\epsilon\{(x-\alpha)^2 - \beta\} + 2dx + r(s+\tau)\}. \tag{30}$$

This differential equation generate the spiking burst for $0.82 < I_{<}3.25$[5]. At this time, the potential function forms single well, and a equilibrium point is covered by b_1 active area that controls the slow oscillation. b_2 active area is placed at beside equilibrium point. That is, the form of the potential and the position of active areas on the potential(Fig.3) are similar to coupled van der Pol system. Increasing the time constant τ, the dynamics of this model change dramatically. The output x is shown in Fig.4(b). One of the features of this output, it has rapid oscillations at the top of the saddle of the output. This type of bursting oscillation called tapered bursting[6] is observed with Burst ID model([3] Fig.6(b)). We make a connection be-

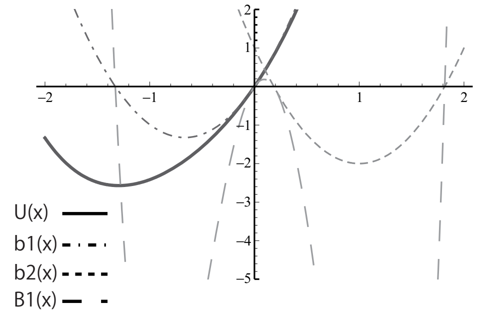

Figure 3: Potential and active area of Hindmarsh rose model where $I = 1.4$, when we are able to observe a regularly bursting oscillation.

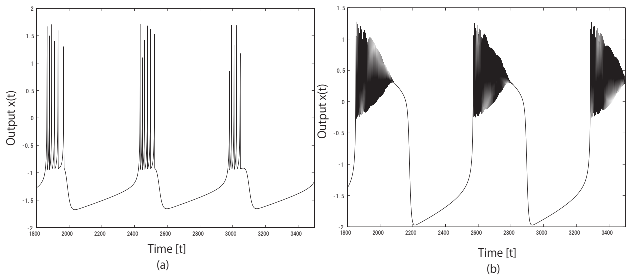

Figure 4: The time series of output x(t) and y(t) for Hindmarsh-Rose type model without any connections, where $\alpha = 1, \beta = 1, d = 5, \theta = 1, s = 4, Z_0 = -1.6, r = 0.001$,(a) $I = 1.4, \tau = 1$, (b) $I = 3.0, \tau_y = 2$.

tween this models in the form of the follows,

$$\frac{d^3x}{dt^3} + b_2(x)\frac{d^2x}{dt^2} + b_1(x,\dot{x})\frac{dx}{dt} = -\frac{r}{\tau_x}(g(x) + z(x) + W(x) - I_x - \theta_x) + \omega y, \quad (31)$$

$$\frac{d^3y}{dt^3} + b_2(y)\frac{d^2y}{dt^2} + b_1(y,\dot{y})\frac{dy}{dt} = -\frac{r}{\tau_y}(g(y) + z(y) + W(y) - I_y - \theta_y) + \omega x. \quad (32)$$

Figure 5 shows time series of the output $x(t)$ and $y(t)$, where $\tau_x = 1.0, \tau_y = 2, I_x = 1.4, I_y = 3, \epsilon = 3.0, \alpha = 1.0, \beta = 1.0, d = 5, s = 4, z_0 = -1.6, \theta = 1, \omega = 0.0005, r_x = 0.001, r_y = 0.001$. Indexes x and y denote each units. This model generate the periodic spiking pulse in stand-alone, if $I_x = 1.4$. We are able to observe regular bursting spikes, the output x and y show two type burst firing oscillate in anti-phase. As we increase the connection strength ω, the period of the bursting oscillations becomes longer, being infinity eventually. That is, the outputs x and y become a resting state. This dynamics is caused by a oscillation of the potential expressed in Fig.3. The potential is forced to oscillate by the bursting output $y(t)$.

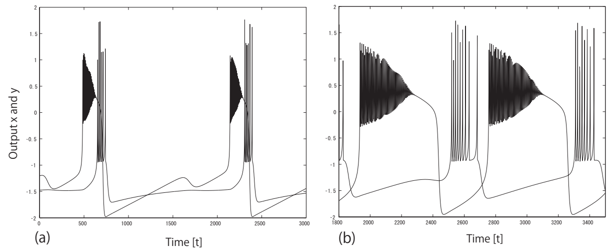

Figure 5: Time series of the output $x(t)$ and $y(t)$ of the coupled bursting oscillators, where (a) $\omega = -0.0005$, (b) $\omega = 0.00058$. It shows two type bursting oscillations in anti-phase of the spiking bursting and tapered bursting. The period of bursting become longer as increasing the parameter ω.

5. Conclusion

We apply our concept of the potential with active areas to interconnected system. The bursting dynamics with undershoot in single well is revealed. This dynamics has relation to the position of active area that controls fast oscillation for the equilibrium point. We figure out the dynamics of the coupled system between bursting oscillators through our concept.

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