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Fluctuations of fuzzy cellular automata around their convergence point

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Abstract—In this paper, we study the oscillating behavior of a class of fuzzy cellular automata as they converge toward their fixed point. We first prove that they all converge to $\frac{1}{2}$, we then describe their dynamics as values approach this point of convergence. In all cases the fluctuations around $\frac{1}{2}$ obey a Boolean rule. We show that for some fuzzy rules the oscillations follow precisely the corresponding Boolean rule itself, while for others they obey a different rule. Finally, we characterize the class of elementary cellular automata that fluctuate according to their corresponding Boolean rule showing that only those studied in this paper display this behavior. These results explain and generalize those of [9].

1. Introduction

Boolean cellular automata (CA) are totally discrete dynamical systems: discrete in time, space, and states. They were introduced by Von Neumann as models of self-organizing/reproducing behaviors [20] and their applications range from ecology to theoretical computer science (e.g., see [3, 14, 21]).

Continuous cellular automata (or Coupled Map Lattices) are discrete in space and time, but continuous in states. They were introduced by Kaneko as simple models exhibiting spatio-temporal chaos, and now have applications in many different areas including fluid dynamics, biology, chemistry, etc. (e.g., [12, 13]).

Fuzzy cellular automata (FCA) are a particular type of continuous cellular automata where the local transition rule is the "fuzzification" of the local rule of a corresponding Boolean cellular automaton in disjunctive normal form[1]. Introduced in [4, 5] to study the impact that state-discretization has on the behavior of these systems, they have been used to investigate the effect of perturbation (e.g. noisy source, computational error, mutation, etc.) on the evolution of Boolean CA [10]. Recently, they have been shown to be useful tools for pattern recognition purposes (e.g., see [15, 16]), and good models for generating images mimicking nature (e.g. [7, 19]). The asymptotic dynamics of elementary FCA (i.e., with dimension and neighbourhood one) have only recently been studied. In quiescent backgrounds, it has been shown that none of them have chaotic dynamics [9, 17, 18]. Circular elementary FCA have been studied experimentally from random initial configurations in [8], while some of the interesting dynamics have recently been proven analytically in [2].

The two models (binary and continuous states) correspond to extreme levels of discretization and the relationship between these levels is an interesting area of investigation. For example, some studies have been done to approximate CML by CA, i.e., to "discretize" some types of CML (e.g., [6]). Similarly, one could start from fuzzy CA and study the change in dynamics while discretizing the state space, eventually reaching the Boolean model. A discretization of the state space is created,for example, when visualizing the space-time diagram of continuous CAs: the continuous interval is in fact divided in k ranges and each is assigned a different colour (the level of discreteness is given by the choice of k). In this regard, a surprising observation was made in the case of elementary rule 90, whose Boolean version has received a lot of attention (e.g., see [11]), where, depending on the level of discretization, the space-time diagram can at times show a dynamics identical to the well known complex Boolean behavior, while at other times a simple convergence. This observation led to the discovery of a very interesting asymptotic phenomenon in fuzzy rule 90 [9]: the dynamics of the fluctuations of fuzzy rule 90 around its convergence point of $\frac{1}{2}$ obey the rule table of the corresponding Boolean rule 90. This opens an interesting new avenue of investigation: the study and understanding of the behavior of a convergent rule in proximity to its point of convergence.

Motivated by this phenomenon, we now study the behavior of *self-averaging* rules, a class of elementary rules which includes rule 90, around their convergence point. Beside proving that they converge to $\frac{1}{2}$, we also describe their dynamics in detail as values approach this point of convergence. We discover that, in all cases, their fluctuations around $\frac{1}{2}$ obey a Boolean rule. In the case of rules 60, 90, 105, and 150 the Boolean rule that describes the fluctuations coincides with the fuzzy rule itself, while for the other self-averaging rules it is different. Finally, we can also show that, of all elementary rules, all and only 60, 90, 105, and 150 fluctuate around $\frac{1}{2}$ obeying their corresponding Boolean rule table. These results generalize the ones of [9]. Due to the lack of space some of the proofs are sketched and some omitted.

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[1]These are not to be confused with a variant of cellular automata, also called fuzzy cellular automata, where the fuzziness refers to the choice of a deterministic local rule (e.g., see [1])

2. Definitions

An elementary circular Boolean cellular automaton consists of a collection of cells arranged in a linear array. Cells have Boolean values and they synchronously update their values according to a local rule applied to their neighbourhood. A configuration $\mathbf{X}^t = (\ldots, x_{i-1}^t, x_i^t, x_{i+1}^t, \ldots)$ is a description of all cell values at a given time t. The neighbourhood of a cell consists of the cell itself and its left and right neighbours, thus the local rule has the form: $g : \{0,1\}^3 \to \{0,1\}$. The local rule g of a Boolean CA is typically given in tabular form by listing the 8 binary tuples corresponding to the possible local configurations a cell can detect in its direct neighbourhood, and mapping each tuple to a Boolean value r_i ($0 \le i \le 7$): $(000, 001, 010, 011, 100, 101, 110, 111) \to (r_0, \cdots, r_7)$. The binary representation $(r_0, \cdots, r_7)$ is often converted into the decimal representation $\sum_i r_i$, and this value is typically used as the "name" of the rule (or rule number). Let us denote by d_i the tuple mapping to r_i, and by $\mathcal{T}_1$ the set of tuples mapping to one. The local rule can also be canonically expressed in *disjunctive normal form* (DNF) as follows:

$$g(v_0, v_1, v_2) = \bigvee_{i=0:7} r_i \bigwedge_{j=0:2} v_j^{d_{i,j}}$$

where d_{ij} is the j-th digit, from left to right of d_i (counting from zero) and v_j^0 (resp. v_j^1) stands for $\neg v_j$ (resp. v_j).

A *fuzzy cellular automaton* (FCA) is a particular continuous cellular automaton where the local rule is obtained by *DNF-fuzzification* of the local rule of a classical Boolean CA. The fuzzification consists of a fuzzy extension of the boolean operators AND, OR, and NOT in the DNF expression of the Boolean rule. Depending on which fuzzy operators are used, different types of fuzzy cellular automata can be defined. Among the various possible choices, we consider the following: $(a \vee b)$ is replaced by $max\{1, (a+b)\}^2$, $(a \wedge b)$ by (ab), and $(\neg a)$ by $(1-a)$ (also indicated by $\bar{a}$). The resulting local rule $f : [0,1]^3 \to [0,1]$ becomes a real function that generalizes the canonical representation of the corresponding Boolean CA:

$$f(v_0, v_1, v_2) = \sum_{i=0:7} r_i \prod_{j=0:2} l(v_j, d_{i,j}) \qquad (1)$$

where $l(a, 0) = 1 - a$ and $l(a, 1) = a$.

Throughout this paper, we will denote local rules of Boolean CA by g_n and their fuzzifications for the corresponding FCA by f_n, where n is the rule number.

A rule is said to *converge* to an homogeneous configuration $(\ldots p, p, p, \ldots, p, p, p, \ldots)$ if, starting from an initial configuration $(\ldots, x_{i-1}^0, x_i^0, x_{i+1}^0 \ldots)$ with $\forall i \;\; x_i^0 \in (0,1)$, we have that $\forall \epsilon > 0 \; \exists T$ such that $\forall t > T$ and $\forall i$: $|x_i^t - x_i^{t+1}| < \epsilon$. In this case, we will say that rule f *converges to* p.

[2]note that, in the case of FCA, $max\{1, (a+b)\} = (a+b)$

In the paper we are interested in the behavior of *self-averaging rules*: a particular class of elementary fuzzy CA. Self-averaging rules can be written as the weighted average of one of their variables as follows: $f(x, y, z) = \gamma x + (1-\gamma)(1-x)$ (analogously for variables y and z). For example, rule 30 can be written as: $((1-y)(1-z)x + ((1-y)z + y(1-z) + yz)(1-x)$ and it is easy to see that, in this case $\gamma = (1-y)(1-z)$.

Table 1 contains all the elementary self-averaging rules where $\bar{x}$ indicates the value $(1-x)$.

Rule	Equation
f_{30}	$(\bar{y}\bar{z})x + (\bar{y}z + y\bar{z} + yz)\bar{x}$
f_{45}	$(\bar{y}z)x + (\bar{y}\bar{z} + y\bar{z} + yz)\bar{x}$
f_{54}	$(\bar{x}\bar{z})y + (\bar{x}z + x\bar{z} + xz)\bar{y}$
f_{57}	$(\bar{x}z)y + (\bar{x}\bar{z} + x\bar{z} + xz)\bar{y}$
f_{60}	$(\bar{x})y + (x)\bar{y}$
f_{90}	$(\bar{x})z + (x)\bar{z}$
f_{105}	$(\bar{x}y + x\bar{y})z + (\bar{x}\bar{y} + xy)\bar{z}$
f_{106}	$(\bar{x}\bar{y} + \bar{x}y + x\bar{y})z + (xy)\bar{z}$
f_{150}	$(\bar{x}\bar{z} + xz)y + (\bar{x}z + x\bar{z})\bar{y}$
f_{154}	$(\bar{x}y + \bar{x}\bar{y} + xy)z + (x\bar{y})\bar{z}$

Table 1: Self-averaging elementary fuzzy CA rules (the rules equivalent under conjugation, reflection, or both are not indicated).

3. Fuctuations around the convergence point

In this section we study the fluctuations of all self-averaging fuzzy rules rules around their convergence point.

3.1. Convergence

We first prove the convergence to $\frac{1}{2}$ of all self-averaging rules.

Lemma 3.1. *Given an initial configuration X^0 and any rule of the form $x_i^{t+1} = \gamma_i^t x_j^t + (1-\gamma_i^t)(1-x_j^t)$ with $j \in \{i-1, i, i+1\}$ and $\gamma_i^t \in (0,1)$ for all i and t. If there exists $0 < \gamma < \frac{1}{2}$ such that $\gamma \le \gamma_i^t \le (1-\gamma)$ then $x_i^t \to \frac{1}{2}$ for all i as $t \to \infty$.*

Proof. Without loss of generality, assume that $x_j^t < \frac{1}{2}$, and let $x_j^t = \frac{1}{2} - \epsilon_j^t$ for some $0 < \epsilon_j^t < \frac{1}{2}$. Then x_i^{t+1} is bounded by $(1-\gamma)(\frac{1}{2}-\epsilon_j^t) + \gamma(\frac{1}{2}+\epsilon_j^t)$ and $\gamma(\frac{1}{2}-\epsilon_j^t) + (1-\gamma)(\frac{1}{2}+\epsilon_j^t)$. Re-arranging, we obtain $|\frac{1}{2} - x_i^{t+1}| < \epsilon_j^t(1-2\gamma)$. Thus, $|\frac{1}{2} - x_i^{t+1}| \to 0$, and $x_i^t \to \frac{1}{2}$. □

Theorem 3.1. *All self-averaging rules converge to $\frac{1}{2}$.*

Proof. The proofs for rules 60, and 90 can be derived directly by applying Lemma 3.1. We now give the details for the proof of convergence for rule 45. Proofs for the remaining rules are analogous.

Recall $f_{45}(x, y, z) = (\bar{y}z)x + (\bar{y}\bar{z} + y\bar{z} + yz)\bar{x}$. Let $\gamma = \min_i\{(\bar{x}_i^0)^2, (x_i^0)^2, 1-(\bar{x}_i^0)^2, 1-(x_i^0)^2\}$. We will show by induction that γ is a bound on the weights as required by Lemma 3.1.

First note that $1-\gamma = \max_i\{(\bar{x}_i^0)^2, (x_i^0)^2, 1-(\bar{x}_i^0)^2, 1-(x_i^0)^2\}$. This follows from the fact that if $\alpha \in \{(\bar{x}_i^0)^2, (x_i^0)^2, 1-(\bar{x}_i^0)^2, 1-(x_i^0)^2\}$, then so is $1-\alpha$. In particular, $1-\gamma$ is in this set. Now let $1-\alpha = \max_i\{(\bar{x}_i^0)^2, (x_i^0)^2, 1-\bar{x}_i^0)^2, 1-(x_i^0)^2\}$ for some α in the set. Then $1-\alpha \geq 1-\gamma$ implies that $\alpha \leq \gamma$. Since γ is the minimum, we must have $\alpha = \gamma$.

We also need to show that $\gamma \leq \bar{x}_i^0 x_{i+1}^0 \leq (1-\gamma)$ for all i. Assume for some i that $\bar{x}_i^0 \leq x_{i+1}^0$. Then $\bar{x}_i^0 x_{i+1}^0 \geq (\bar{x}_i^0)^2 \geq \gamma$. Also, $\bar{x}_i^0 x_{i+1}^0 \leq (x_{i+1}^0)^2 \leq (1-\gamma)$. The argument is similar when $\bar{x}_i^0 \geq x_{i+1}^0$.

Now assume that at time t, for all i, $\gamma \leq \bar{x}_i^t x_{i+1}^t \leq (1-\gamma)$. Then from the proof of Lemma 3.1, $|x_i^{t+1} - \frac{1}{2}| < |x_i^t - \frac{1}{2}| \;\; \forall i$. Since either x_i^t or $\bar{x}_i^t$ must be less than or equal to $\frac{1}{2}$, without loss of generality, consider $x_i^t \leq \frac{1}{2}$. Then $(x_i^t)^2 < 1-(x_i^t)^2$, and $(x_i^t)^2 \leq (x_i^{t+1})^2$. Hence, $(x_i^{t+1})^2$ and $1-(x_i^{t+1})^2$ are greater than γ which implies in turn that they are both less than $1-\gamma$. We see that $\gamma \leq \min_i\{(\bar{x}_i^{t+1})^2, (x_i^{t+1})^2, 1-\bar{x}_i^{t+1})^2, 1-(x_i^{t+1})^2\}$ and $1-\gamma \geq \max_i\{(\bar{x}_i^{t+1})^2, (x_i^{t+1})^2, 1-\bar{x}_i^{t+1})^2, 1-(x_i^{t+1})^2\}$. As before, we have $\gamma \leq \bar{x}_i^{t+1} x_{i+1}^{t+1} \leq (1-\gamma)$.

So Lemma 3.1 applies, and $\forall i$, $x_i^t \to \frac{1}{2}$ as $t \to \infty$. $\square$

3.2. Auto-Fluctuations

We now study the self-averaging fuzzy rules whose behavior around $\frac{1}{2}$ obeys the corresponding Boolean rule. Before starting the analysis, we introduce a technical lemma.

Lemma 3.2. *$\alpha\beta + \bar{\alpha}\bar{\beta}$ is greater than $\frac{1}{2}$ if and only if both β and α are greater than $\frac{1}{2}$ or both are smaller.*

Proof. Assume $\alpha\beta + (1-\alpha)(1-\beta) > \frac{1}{2}$. Rearranging we obtain: $(2\alpha-1)\beta > \frac{1}{2}(2\alpha-1)$. If $\alpha > \frac{1}{2}$ then $(2\alpha-1) > 0$, and $\beta > \frac{1}{2}$. Otherwise, if $\alpha < \frac{1}{2}$ then $\beta < \frac{1}{2}$. $\square$

x	y	z	$f_{60}(x,y,z)$	x	y	z	$g_{60}(x,y,z)$
$<$	$<$	$<$	$<$	0	0	0	0
$<$	$<$	$>$	$<$	0	0	1	0
$<$	$>$	$<$	$>$	0	1	0	1
$<$	$>$	$>$	$>$	0	1	1	1
$>$	$<$	$<$	$>$	1	0	0	1
$>$	$<$	$>$	$>$	1	0	1	1
$>$	$>$	$<$	$<$	1	1	0	0
$>$	$>$	$>$	$<$	1	1	1	0

Table 2: Rule 60: fluctuations of the fuzzy rule around $\frac{1}{2}$ (left); Boolean table (right). Symbols $>$ and $<$ respectively indicate values greater than or smaller than $\frac{1}{2}$.

Theorem 3.2. *The fluctuations of fuzzy rules f_{60}, f_{90}, f_{105}, and f_{150} around their point of convergence of $\frac{1}{2}$ obey their corresponding Boolean rule.*

x	y	z	$f_{150}(x,y,z)$		x	y	z	$g_{150}(x,y,z)$
$<$	$<$	$<$	$<$		0	0	0	0
$<$	$<$	$>$	$>$		0	0	1	1
$<$	$>$	$<$	$>$		0	1	0	1
$<$	$>$	$>$	$<$		0	1	1	0
$>$	$<$	$<$	$>$		1	0	0	1
$>$	$<$	$>$	$<$		1	0	1	0
$>$	$>$	$<$	$<$		1	1	0	0
$>$	$>$	$>$	$>$		1	1	1	1

Table 3: Rule 150: fluctuations of the fuzzy rule around $\frac{1}{2}$ (left); Boolean table (right).

Proof. By Theorem 3.1 we know that all these rules converge to $\frac{1}{2}$, we now derive their dynamics around it. Rule 60 has the following analytical form: $f_{60}(x,y,z) = (\bar{x})y + (x)\bar{y}$. By Lemma 3.2 letting $\alpha = x$ and $\beta = y$, we have:

$$f_{60}(x,y,z) > \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } y > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y < \frac{1}{2} \end{cases}$$

$$f_{60}(x,y,z) < \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } y < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y > \frac{1}{2} \end{cases}$$

which can be written around $\frac{1}{2}$ as in Table 2 (left), and which coincides with Boolean rule 60 (right) where 0 corresponds to $<$ and 1 to $>$. The proof for rule 90 is identical, letting $\alpha = x$ and $\beta = z$.

Consider now rule 150: $f_{150}(x,y,z) = (\bar{x}\bar{z} + xz)y + (\bar{x}z + x\bar{z})\bar{y}$. We apply Lemma 3.2 to this rule letting $\alpha = (\bar{x}z + x\bar{z})$ and $\beta = y$ so that

$$f_{150}(x,y,z) > \frac{1}{2} \quad \text{if} \quad \begin{cases} \alpha < \frac{1}{2} \text{ and } y > \frac{1}{2} \\ \alpha > \frac{1}{2} \text{ and } y < \frac{1}{2} \end{cases}$$

$$f_{150}(x,y,z) < \frac{1}{2} \quad \text{if} \quad \begin{cases} \alpha < \frac{1}{2} \text{ and } y < \frac{1}{2} \\ \alpha > \frac{1}{2} \text{ and } y > \frac{1}{2} \end{cases} .$$

Then we apply the lemma directly to α:

$$\alpha > \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } z < \frac{1}{2} \end{cases}$$

$$\alpha < \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } z > \frac{1}{2} \end{cases} . .$$

Combining these results, we obtain:

$$f_{150}(x,y,z) > \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x < \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z > \frac{1}{2} \end{cases}$$

$$f_{150}(x,y,z) < \frac{1}{2} \quad \text{if} \quad \begin{cases} x < \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z < \frac{1}{2} \\ x < \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y < \frac{1}{2} \text{ and } z > \frac{1}{2} \\ x > \frac{1}{2} \text{ and } y > \frac{1}{2} \text{ and } z < \frac{1}{2} \end{cases}$$

which again describes a fluctuation table around $\frac{1}{2}$ that coincides with the corresponding Boolean rule table 3 of rule 150. The proof for rule 105 follows in the same way letting $\alpha = (\bar{x}\bar{y} + xy)$ and $\beta = z$ when applying Lemma 3.2. $\square$

3.3. Other fluctuations

Analysis of the remaining self-averaging rules is a little more complex.

Theorem 3.3. *As fuzzy rules $f_{45}, f_{57}, f_{106}, f_{154}, f_{39}$ and f_{54} converge to $\frac{1}{2}$, their fluctuations around $\frac{1}{2}$ obey the Boolean rules listed here: f_{45} obeys g_{15}; f_{57} obeys g_{51}; f_{106} obeys g_{170}; f_{154} obeys g_{170}; f_{30} obeys g_{15}; f_{54} obeys g_{51}.*

Proof. By Theorem 3.1, we know that all these rules converge to $\frac{1}{2}$. We will give the proof of the fluctuations in detail for rule 45. Similar arguments show the behaviours of the other rules. We apply Lemma 3.2 to rule 45: $f_{45}(x, y, z) > \frac{1}{2}$ when $x > \frac{1}{2}$ and $\bar{y}z = (1-y)z > \frac{1}{2}$ or when $x < \frac{1}{2}$ and $(1-y)z < \frac{1}{2}$. We begin by assuming that $x > \frac{1}{2}$. Then $(1-y)z > \frac{1}{2}$ if $(1-y) > \frac{1}{2}$ (so $y < \frac{1}{2}$) and $z > \frac{1}{2(1-y)}$. As $y \to \frac{1}{2}$, $\frac{1}{2(1-y)} \to 1$ so that when y and z are close enough to $\frac{1}{2}$, z is never greater than $\frac{1}{2(1-y)}$. More precisely, when $\frac{1}{2} - \frac{(\sqrt{2}-1)}{2} < y, z < \frac{1}{2} + \frac{(\sqrt{2}-1)}{2}$, $\frac{1}{2(1-y)} > \frac{1}{2} + \frac{(\sqrt{2}-1)}{2} > z$, hence $f_{45}(x, y, z) < \frac{1}{2}$.

If instead $x < \frac{1}{2}$, $f_{45}(x, y, z) > \frac{1}{2}$ if $(1-y)z < \frac{1}{2}$ which is true whenever $y > \frac{1}{2}$ or $z < \frac{1}{2(1-y)}$. As before, as $y \to \frac{1}{2}$, $\frac{1}{2(1-y)} \to 1$ so that this condition is always true on the interval given above.

In other words, the fluctuations of the fuzzy rule resemble Boolean rule 15 (see Table 4).

x	y	z	f_{45}	x	y	z	g_{15}
$<$	$<$	$<$	$>$	0	0	0	1
$<$	$<$	$>$	$>$	0	0	1	1
$<$	$>$	$<$	$>$	0	1	0	1
$<$	$>$	$>$	$>$	0	1	1	1
$>$	$<$	$<$	$<$	1	0	0	0
$>$	$<$	$>$	$<$	1	0	1	0
$>$	$>$	$<$	$<$	1	1	0	0
$>$	$>$	$>$	$<$	1	1	1	0

Table 4: Fuzzy rule 45 in proximity of $\frac{1}{2}$: fluctuations around $\frac{1}{2}$ (left); corresponding Boolean behavior coinciding with rule 15.

□

4. Characterization

We can actually show that, up to equivalence, the self-averaging rules of Section 3.2 are the only elementary rules displaying the auto-fluctuating behavior described in this paper. In fact, it can be shown by tedious, exhaustive analysis that only self-averaging rules converge to $\frac{1}{2}$.

Theorem 4.1. *All and only rules f_{60}, f_{90}, f_{105}, and f_{150} are elementary rules which fluctuate around their point of convergence obeying their corresponding Boolean rule.*

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