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Existence of discrete breathers in discrete nonlinear Schrödinger equations with non-weak couplings

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Abstract—Discrete breathers are spatially localized periodic solutions in nonlinear discrete dynamical systems. The anti-integrable limit is defined for the discrete nonlinear Schrödinger equation as the limit of vanishing couplings. There are infinitely many trivial discrete breathers in this limit, each of which consists of a finite number of excited sites. The existence of discrete breathers continued from them has been proved for sufficiently weak couplings. In this paper, we focus on the case of non-weak couplings and present existence theorems of discrete breathers.

1. Introduction

Spatially localized excitations in nonlinear space-discrete dynamical systems have attracted great interest since the seminal work by Takeno *et al.* [1]. The localized modes are called *discrete breathers* (DBs) or *intrinsic localized modes*. The DBs are time-periodic and spatially localized solutions of the equations of motion. Considerable progress has been achieved in understanding the nature of DB so far (e.g., [2, 3, 4, 5]).

The discrete nonlinear Schrödinger equation (DNLSE) is one of the fundamental lattice models (e.g., [6, 7]), which appears in various contexts of physics. The DNLSE is the system of differential equations

$$i\frac{d\psi_n}{dt} + \kappa(\Delta\psi)_n + \gamma|\psi_n|^2\psi_n = 0, \qquad (1)$$

where $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, $\psi_n \in \mathbb{C}$, $\kappa \geq 0$, $\gamma = \pm 1$, and Δ is the discrete Laplacian on a cubic d-dimensional lattice,

$$(\Delta\psi)_n = \sum_{j=1}^{d}\left(\psi_{n+e_j} + \psi_{n-e_j} - 2\psi_n\right), \qquad (2)$$

and $\{e_1, \dots, e_d\}$ are the standard unit vectors on $\mathbb{Z}^d$. Two cases of $\gamma = +1$ and -1 are called the *focusing* and *defocusing* cases, respectively.

In this study, we address the existence of DB solutions to Eq. (1) of the form

$$\psi_n(t) = \phi_n \exp(i\omega t), \qquad (3)$$

where ϕ_n is the time-independent amplitude of site n and $\omega \in \mathbb{R}$ is the frequency. We confine our attention to the solutions such that ϕ_n is real and satisfies $|\phi_n| \to 0$ as $|n| \to +\infty$, where $|n| = \sum_{j=1}^{d} |n_j|$. These DB solutions are sometimes called *bright discrete solitons* in the context of DNLSE.

The existence of bright discrete solitons has been investigated by several authors using different mathematical methods [8, 9, 10, 11, 12, 13]. The *anti-integrable* (AI) limit is a useful concept for proving the existence of DBs. The AI limit was originally introduced by Aubry to study chaotic trajectories of the standard map [14] and then extended to study DBs in nonlinear lattices [15]. The AI limit of the DNLSE is defined by the limit of $\kappa = 0$ in Eq. (1). The system has infinitely many trivial localized periodic solutions such that a finite number of sites are excited with $\phi_n \neq 0$ and the others are at rest with $\phi_n = 0$, provided that ω is fixed. We call these trivial solutions the *anti-integrable solutions*. The existence of bright discrete solitons with the ω has been proved by continuing the AI solutions for small coupling constant, using the implicit function theorem [15]. The continuation is possible up to some positive value κ_c. However, no explicit lower bound for κ_c has been obtained except for the case of $d = 1$ [12]. For $d \geq 2$, the continuation has been proved only for sufficiently weak couplings. In this sense, the existence of bright discrete solitons continued from AI solutions has not been fully proved for the DNLSE with non-weak couplings.

In this paper, for any dimension d, we explicitly give a non-small range of the coupling constant and prove that the bright discrete soliton uniquely continued from an arbitrary AI solution exists over the range. In the one-dimensional case, our estimation is much improved than that obtained in Ref. [12]. The present approach uses Banach's fixed point theorem, and it may apply for different types of localized periodic solutions such as *dark discrete solitons* near the AI limit in the DNLSE.

2. Stationary DNLSE and anti-integrable limit

The bright discrete solitons can be sought by using the steady-state ansatz Eq. (3). Under substitution of the ansatz into Eq. (1), the amplitudes $\phi_n \in \mathbb{R}$ are determined by the set of algebraic equations

$$-\omega\phi_n + \kappa(\Delta\phi)_n + \gamma\phi_n^3 = 0, \quad n \in \mathbb{Z}^d, \qquad (4)$$

and the condition of spatial localization $\phi_n \to 0$ as $|n| \to +\infty$, where $|n| = \sum_{j=1}^{d} |n_j|$. We call Eq. (4) the *stationary DNLSE*.

The phonon band of Eq. (1) is given by $-4d\kappa \le \omega \le 0$. The bright discrete solitons are expected to exist for ω outside the phonon band. Solutions of Eq. (4) for $\gamma = +1$, $\omega > 0$ and those for $\gamma = -1$, $\omega < -4d\kappa$ relate with each other as follows: if $\{\phi_n\}_{n\in\mathbb{Z}^d}$ is a solution of Eq. (4) for $\gamma = +1$ and $\omega = \omega_0 > 0$, then $\{(-1)^{|n|}\phi_n\}_{n\in\mathbb{Z}^d}$ is a solution of Eq. (4) for $\gamma = -1$ and $\omega = -4d\kappa - \omega_0$. This one-to-one correspondence implies that it is enough to consider only one of the focusing and defocusing cases. Let us consider the focusing case $\gamma = +1$. It is known that the stationary DNLSE (4) has no nonzero localized solution when $\omega < -4d\kappa$ [7]. Therefore, we assume $\gamma = +1$ and $\omega > 0$ in this paper.

In order to transform Eq. (4) into a simpler form, let us introduce the new variables $\{u_n\}_{n\in\mathbb{Z}^d}$ defined by $\phi_n = \sqrt{\mu}\, u_n$, where $\mu = \omega + 2d\kappa$. If we use $\{u_n\}_{n\in\mathbb{Z}^d}$ and assume $\gamma = +1$, the stationary DNLSE (4) can be rewritten as

$$u_n - u_n^3 = \varepsilon \sum_{j=1}^{d} \left(u_{n+e_j} + u_{n-e_j} \right), \quad n \in \mathbb{Z}^d, \tag{5}$$

where the parameter ε is defined by $\varepsilon = \kappa/\mu$. Hereafter, equation (5) is used for our study instead of Eq. (4). The AI limit of Eq. (5) is defined as the limit of $\varepsilon = 0$. When $\varepsilon = 0$, equation (5) has an infinite number of AI solutions given by

$$u_n = \sigma_n, \quad \sigma_n \in \{0, \pm 1\}. \tag{6}$$

The site amplitudes u_n are independent of each other, and each u_n can take one of the three values 0 and ± 1. Thus any AI solution can be coded by an infinite sequence $\sigma \equiv \{\sigma_n\}_{n\in\mathbb{Z}^d}$, which is called the *coding sequence*. Let $\mathcal{S}$ be the set of coding sequences σ which have only a finite number of nonzero elements, i.e., $\mathcal{S} = \{\sigma;\ \sum |\sigma_n| < \infty\}$. Each $\sigma \in \mathcal{S}$ gives a localized AI solution.

It is known that all the AI solutions can be continued for sufficiently small ε by the implicit function theorem [15]. The purpose of this study is to show that the continuation is possible for non-small ε, using Banach's fixed point theorem. Let $l^\infty(\mathbb{Z}^d)$ be the Banach space of real-valued bounded sequences $u = \{u_n\}_{n\in\mathbb{Z}^d}$ endowed with the norm $\|u\| = \sup_{n\in\mathbb{Z}^d} |u_n|$:

$$l^\infty(\mathbb{Z}^d) = \left\{ u\,;\ \|u\| = \sup_n |u_n| < +\infty \right\}. \tag{7}$$

We consider Eq. (5) as a nonlinear equation in the space $l^\infty(\mathbb{Z}^d)$.

3. Main results

Given $\sigma \in \mathcal{S}$, let $A(\sigma)$ be the set of indices for nonzero elements of σ, i.e., $A(\sigma) = \{n;\ \sigma_n \neq 0\} \subset \mathbb{Z}^d$. Our theorem for the existence of bright discrete solitons in the d-dimensional DNLSE is stated as follows.

Theorem 1. *Let ε_0, c_0, and r_0 be the constants given by*

$$\varepsilon_0 = \frac{9\sqrt{65} - 71}{16d}, \quad c_0 = \frac{\sqrt{65} - 7}{4}, \quad r_0 = \frac{1}{10}.$$

For any $\sigma \in \mathcal{S}$ and $\varepsilon \in [0, \varepsilon_0)$, there exists a unique family of solutions $\{u_n(\varepsilon)\}_{n\in\mathbb{Z}^d}$ of Eq. (5) such that it is continuous with respect to ε and $u_n(0) = \sigma_n$, $n \in \mathbb{Z}^d$. For each $\varepsilon \in [0, \varepsilon_0)$, the solution $\{u_n(\varepsilon)\}_{n\in\mathbb{Z}^d}$ satisfies

$$|u_n(\varepsilon) - \sigma_n| \le \begin{cases} c & \text{if } |n| \le m+1, \\ c\, r^{|n|-m-1} & \text{otherwise}, \end{cases} \tag{8}$$

with $c = c_0\, \varepsilon/\varepsilon_0$, $r = r_0\, \varepsilon/\varepsilon_0$, and $m = \max_{n\in A(\sigma)} |n|$.

Remark 1. By definition of m, $\sigma_n = 0$ when $|n| > m$. Thus the second inequality in Eq. (8) reduces to $|u_n(\varepsilon)| \le c\, r^{|n|-m-1}$. This indicates that the amplitude $u_n(\varepsilon)$ decays exponentially as $|n| \to \infty$ since $r < 1$.

In Theorem 1, the AI solution $\{\sigma_n\}_{n\in\mathbb{Z}^d}$ is used as an approximate solution for Eq. (5) with nonzero ε. It is possible to improve the range of ε in which the existence of solution $\{u_n(\varepsilon)\}_{n\in\mathbb{Z}^d}$ is guaranteed if a better approximation is used. It is rather easy to compute a better approximation in the one-dimensional case, although it is cumbersome for higher-dimensional cases.

Consider the case of $d = 1$ and suppose that $\sigma \in \mathcal{S}$ has nonzero elements only for $n = n_1, \ldots, n_m$, i.e., $A(\sigma) = \{n_1, \ldots, n_m\} \subset \mathbb{Z}$. An improved approximation can be obtained as follows:

$$u_n^*(\varepsilon) = \begin{cases} \sigma_{n_1} \varepsilon^{n_1 - n} & \text{if } n < n_1, \\ \sigma_n + \chi(\sigma_n)(\sigma_{n+1} + \sigma_{n-1})\varepsilon & \text{if } n_1 \le n \le n_m, \\ \sigma_{n_m} \varepsilon^{n - n_m} & \text{if } n > n_m, \end{cases} \tag{9}$$

where $\chi(q)$ is the function defined by $\chi(q) = (3\delta_{q,0} - 1)/2$ and $\delta_{q,0}$ is Kronecker's delta. Using this approximate solution, we can obtain the following theorem.

Theorem 2. *Suppose that $d = 1$, $\sigma \in \mathcal{S}$, and $A(\sigma) = \{n_1, \ldots, n_m\}$. Let $\{u_n^*\}_{n\in\mathbb{Z}}$ be an approximate solution given by Eq. (9). Let $\varepsilon_0 = 0.1457$, $c_0 = 0.16$, and $r_0 = 0.3$. Then there exists a unique family of solutions $\{u_n(\varepsilon)\}_{n\in\mathbb{Z}}$ of Eq. (5) for $\varepsilon \in [0, \varepsilon_0]$ such that it is continuous with respect to ε and $u_n(0) = \sigma_n$, $n \in \mathbb{Z}$. For each $\varepsilon \in [0, \varepsilon_0]$, the solution $\{u_n(\varepsilon)\}_{n\in\mathbb{Z}}$ satisfies*

$$|u_n(\varepsilon) - u_n^*(\varepsilon)| \le \begin{cases} c\, r^{n_1 - n} & \text{if } n < n_1, \\ c & \text{if } n_1 \le n \le n_m, \\ c\, r^{n - n_m} & \text{if } n > n_m, \end{cases} \tag{10}$$

with $c = c_0\, \varepsilon/\varepsilon_0$ and $r = r_0\, \varepsilon/\varepsilon_0$.

Remark 2. The solution $u_n(\varepsilon)$ decays exponentially as $n \to \pm\infty$ since $|u_n(\varepsilon)| \le (1+c)r^{n_1-n}$ (resp. $(1+c)r^{n-n_m}$) holds for $n < n_1$ (resp. $n > n_m$) from Eqs. (9) and (10).

Remark 3. An essential parameter of Eq. (4) is $\alpha \equiv \kappa/\omega$

when $\gamma = +1$, and $\alpha = 0$ defines the AI limit. It was proved in Ref. [12] that any AI solution can be continued at least up to $\alpha = (3\sqrt{3} - 1)/52 \simeq 0.0807$. The parameter α relates with ε as $\alpha = \varepsilon/(1 - 2d\varepsilon)$. Our estimation $\varepsilon_0 = 0.1457$ corresponds to $\alpha = 0.2056\cdots$, which is much improved and close to the boundary of continuation $\alpha \simeq 0.28958$ obtained numerically in Ref. [12].

4. Formulation of a fixed point problem

We formulate the problem of solving Eq. (5) as a fixed point problem. Consider Eq. (5) in the space $l^\infty(\mathbb{Z}^d)$. Let $a = \{a_n\}_{n\in\mathbb{Z}^d} \in l^\infty(\mathbb{Z}^d)$, and define a new variable $x = \{x_n\}_{n\in\mathbb{Z}^d}$ by

$$u_n = a_n + x_n, \quad n \in \mathbb{Z}^d. \tag{11}$$

The sequence $\{a_n\}_{n\in\mathbb{Z}^d}$ is an approximate solution of Eq. (5), and it is chosen such that $1 - 3a_n^3 \neq 0$ for all $n \in \mathbb{Z}^d$. If we use Eq. (11), we can rewrite Eq. (5) as follows:

$$x_n = \frac{1}{1 - 3a_n^2}\left[\varepsilon \sum_{j=1}^{d}\left(x_{n+e_j} + x_{n-e_j}\right) + 3a_n x_n^2 + x_n^3 + R_n(a)\right], \tag{12}$$

where $n \in \mathbb{Z}^d$ and $R_n(a)$ is the residual given by

$$R_n(a) = \varepsilon \sum_{j=1}^{d}\left(a_{n+e_j} + a_{n-e_j}\right) - a_n + a_n^3. \tag{13}$$

The right hand side of Eq. (12) defines the nonlinear map $F_\varepsilon : l^\infty(\mathbb{Z}^d)\times\mathbb{R} \to l^\infty(\mathbb{Z}^d)$, $(x, \varepsilon) \mapsto F_\varepsilon x$, which depends on the parameter ε. In this notation, equation (12) is written in the simple form

$$x = F_\varepsilon x, \quad x \in l^\infty(\mathbb{Z}^d). \tag{14}$$

This shows that a solution of Eq. (12) is regarded as a fixed point of the map F_ε.

In the present formulation of fixed point problem, we did not remove all the terms that are linear in x from the right hand side of Eq. (12), but kept the linear terms having the small coefficient ε. The explicit form of the map F_ε is available due to this formulation, and thus it is possible to precisely evaluate $F_\varepsilon x$. It should be noted that the map F_ε is still contractive because of smallness of ε.

We are interested in solutions of Eq. (14) that are exponentially localized in space. Therefore, it is necessary to define an appropriate subset of $l^\infty(\mathbb{Z}^d)$ for solving Eq. (14), which guarantees the localization property of the obtained solutions.

Let $\sigma \in \mathcal{S}$, $A(\sigma) = \{\, n;\ \sigma_n \neq 0\,\}$, and $m = \max_{n\in A(\sigma)} |n|$. Define a closed convex subset $B_\sigma(c, r) \subset l^\infty(\mathbb{Z}^d)$ as follows:

$$B_\sigma(c,r) = \Big\{ x\,;\ |x_n| \le c \ \text{ if } |n| \le m+1, \\ |x_n| \le c r^{|n|-m-1} \ \text{ otherwise} \Big\},$$

where $c > 0$ and $0 < r < 1$ are the parameters specifying $B_\sigma(c, r)$. We use this subset to prove Theorem 1.

In order to prove Theorem 2, we use a slightly different subset. Consider the case of $d = 1$, and let $\sigma \in \mathcal{S}$ and $A(\sigma) = \{n_1, \ldots, n_m\}$. We define a closed convex subset $B^1_\sigma(c, r) \subset l^\infty(\mathbb{Z})$ as follows:

$$B^1_\sigma(c,r) = \Big\{ x\,;\ |x_n| \le c r^{n_1-n} \text{ if } n < n_1, \\ |x_n| \le c \text{ if } n_1 \le n \le n_m,\ |x_n| \le c r^{n-n_m} \text{ if } n > n_m \Big\},$$

where $c > 0$ and $0 < r < 1$ are the parameters specifying $B^1_\sigma(c, r)$.

5. Outline of proofs

A basic tool of our proofs of Theorems 1 and 2 is the following Banach's fixed point theorem for a parameter-dependent map. We sketch the proof of Theorem 1 in this section. Theorem 2 can be proved in a similar manner, using $\{u^*_n\}_{n\in\mathbb{Z}}$ and $B^1_\sigma(c, r)$.

Theorem 3. (e.g., [16]) *Let B be a nonempty closed convex set in a Banach space X and $\Lambda \subset \mathbb{R}$ be an interval. Let $F_\lambda : X \times \Lambda \to X$ be an operator which is dependent on a parameter $\lambda \in \Lambda$ and continuously differentiable with respect to x for each $\lambda \in \Lambda$. Suppose that the operator F_λ satisfies the conditions*

1. *for each $\lambda \in \Lambda$, B is mapped into B by F_λ;*
2. *there exists a λ-independent constant $K \in [0, 1)$ such that $\|DF_\lambda(x)\| \le K$ for all $x \in B$ and all $\lambda \in \Lambda$, where $DF_\lambda(x)$ is the Fréchet derivative of F_λ with respect to x;*
3. *for a fixed $\lambda_0 \in \Lambda$, and for all $x \in B$, $\lim_{\lambda\to\lambda_0} F_\lambda x = F_{\lambda_0} x$.*

Then, the equation $x = F_\lambda x$ has a unique solution $x(\lambda) \in B$ for each $\lambda \in \Lambda$, and $x(\lambda)$ is continuous with respect to λ.

5.1. Proof of Theorem 1

We can prove the following two lemmas to apply the Banach's fixed point theorem.

Lemma 1. *Let $\varepsilon \ge 0$, $c > 0$, and $0 < r < 1$. Suppose that the following inequalities are satisfied:*

$$(a) \quad c - c^3 - 2\varepsilon d(1 + c) \ge 0,$$
$$(b) \quad 2c - 3c^2 - c^3 - 2\varepsilon d(1 + c) \ge 0,$$
$$(c) \quad \varepsilon d\left(r + r^{-1}\right) + c^2 r^2 \le 1.$$

Then $B_\sigma(c, r)$ is mapped into $B_\sigma(c, r)$ by F_ε with $a_n = \sigma_n$, $n \in \mathbb{Z}^d$.

Lemma 2. *Let $\varepsilon \ge 0$, $c > 0$, and $0 < r < 1$. Given $K > 0$, suppose that the following inequalities are satisfied:*

$$(d) \quad 2\varepsilon d + 3c^2 \le K,$$
$$(e) \quad \varepsilon d + 3c + \frac{3}{2}c^2 \le K.$$

Then the Fréchet derivative of F_ε *with* $a_n = \sigma_n$, $n \in \mathbb{Z}^d$ *satisfies* $\|DF_\varepsilon(x)\| \le K$ *for all* $x \in B_\sigma(c, r)$.

Let us consider the subset $B_\sigma(c_0, r_0)$, where c_0 and r_0 are the constants given is Theorem 1. Obviously, $B_\sigma(c_0, r_0)$ is a nonempty closed convex subset of Banach space $X \equiv l^\infty(\mathbb{Z}^d)$. Fix an arbitrary $\varepsilon'_0 \in (0, \varepsilon_0)$ and assume $\varepsilon \in I \equiv [0, \varepsilon'_0]$. We consider the map $F_\varepsilon : X \times I \to X$ with the approximate solution $a_n = \sigma_n$, $n \in \mathbb{Z}^d$, and check the conditions (i)-(iii) to apply Theorem 3.

The left hand side of (a) equals to $(\sqrt{65} - 3)(9\sqrt{65} - 71 - 16\varepsilon d)/32$ for $(c, r) = (c_0, r_0)$, and this is larger than zero for $\varepsilon \in I$. Similarly, we have $(\sqrt{65} - 3)(5\sqrt{65} - 27 - 112\varepsilon d)/224 > 0$ and $(57 - 7\sqrt{65})/800 + 101\varepsilon d/10 < 1$ for the left hand sides of (b) and (c) when $\varepsilon \in I$, respectively. By Lemma 1, we have $F_\varepsilon(B_\sigma(c_0, r_0)) \subset B_\sigma(c_0, r_0)$ for all $\varepsilon \in I$.

Substituting $c = c_0$, we obtain $2\varepsilon d + 3c_0^2 \le 2\varepsilon'_0 d + (171 - 21\sqrt{65})/8$ and $\varepsilon d + 3c_0 + 3c_0^2/2 \le \varepsilon'_0 d - (9\sqrt{65} - 71)/16 + 1$ for the left hand sides of (d) and (e) when $\varepsilon \in I$, respectively. Let $h(\varepsilon'_0) = \max\{2\varepsilon'_0 d + (171 - 21\sqrt{65})/8, \varepsilon'_0 d - (9\sqrt{65} - 71)/16 + 1\}$. Since $h(\varepsilon'_0) < 1$, there exists an ε-independent constant $K \in [h(\varepsilon'_0), 1)$ such that both (d) and (e) hold for c_0 and $\varepsilon \in I$. By Lemma 2, we have $\|DF_\varepsilon(x)\| \le K < 1$ for all $x \in B_\sigma(c_0, r_0)$ and all $\varepsilon \in I$.

The conditions (i) and (ii) are satisfied as shown above, and (iii) is obvious. By Theorem 3, the equation $x = F_\varepsilon x$ has a unique family of solutions $x(\varepsilon) \in B_\sigma(c_0, r_0)$ for $\varepsilon \in I$ which is continuous with respect to ε. Since $\varepsilon'_0 \in (0, \varepsilon_0)$ is arbitrary, $x(\varepsilon)$ exists for all $\varepsilon \in [0, \varepsilon_0)$. The solution of Eq. (5) is obtained as $u(\varepsilon) = \sigma + x(\varepsilon)$.

Let $c(\varepsilon) = c_0\varepsilon/\varepsilon_0$ and $r(\varepsilon) = r_0\varepsilon/\varepsilon_0$. The left hand sides of (a)-(e) are polynomials of ε under substitution of $(c, r) = (c(\varepsilon), r(\varepsilon))$. Fix an arbitrary $\varepsilon \in [0, \varepsilon_0)$. It can be checked that the inequalities (a)-(c) hold, and then $F_\varepsilon(B_\sigma(c(\varepsilon), r(\varepsilon)) \subset B_\sigma(c(\varepsilon), r(\varepsilon))$ follows by Lemma 1. It also can be checked that the left hand sides of (d) and (e) are strictly less than unity. By Lemma 2, there exists $K \in [0, 1)$ such that $\|DF_\varepsilon(x)\| \le K < 1$ for all $x \in B_\sigma(c(\varepsilon), r(\varepsilon))$. Thus the equation $x = F_\varepsilon x$ for the fixed value of ε has a unique solution x^* in $B_\sigma(c(\varepsilon), r(\varepsilon)) \subset B_\sigma(c_0, r_0)$, by Banach's fixed point theorem. This solution x^* coincides with $x(\varepsilon)$ because of the uniqueness of $x(\varepsilon)$ in $B_\sigma(c_0, r_0)$. Equation (8) follows from $x(\varepsilon) \in B_\sigma(c(\varepsilon), r(\varepsilon))$. □

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