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Symbolic Dynamics of Some Bernoulli-shift Cellular Automata Rules

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Abstract—In our recent study, the dynamics of some elementary cellular automata rules are investigated in the bi-infinite symbolic sequence space. These rules, as members of the Wolfram class II and Chua's topologically-distinct Bernoulli-shift rules, which were believed to be simply periodic before, actually display rich and complex dynamics. Rules 2 and 35, for example, to be discussed in this paper, are chaotic in the sense of Li-Yorke and/or Devaney; they are topologically transitive or topologically mixing; they have positive topological entropies; and they show a catalog of gliders and glider collisions.

1. Introduction

Cellular Automata (CA), formally introduced by Neumann in the early 1950's, are a class of spatially and temporally discrete, deterministic mathematical systems characterized by local interactions and an inherently parallel from evolution [1]. Twenty years later, Conway proposed his now-famous game of life [2]. From a theoretical point of view, it is interesting because it has the power of a universal Turing machine: anything that can be computed algorithmically can be computed within this game of life. Mathematical theory of CA was further developed by Hedlund in 1969 [3], who viewed one-dimensional CA (1D CA) in the context of symbolic dynamics as homomorphisms of the full shift. His main research is not directly related to the characterizations of surjective and open CA but with the current problems in symbolic dynamics. In the early 1980's, Wolfram carried out intensive research on dynamical and computational aspects of CA [4-6]. He classified the 256 elementary cellular automata (ECA) informally into four classes using dynamical concepts like periodicity, stability, chaos and complexity. In 2002, Wolfram introduced his monumental work *A New Kind of Science* [7]. Based on this work, Chua *et al.* provided a nonlinear dynamics perspective to Wolfram's empirical observations from the viewpoint of mathematical analysis using tools like characteristic function, forward time-τ map, basin tree diagram and Isle-of-Eden digraph [8-11].

Although there are 256 ECA rules, only 88 rules are globally independent from each other. These 88 rules are organized into four groups with distinct qualitative dynamics, corresponding to random initial configurations: period-k rules ($k = 1, 2, 3, 6$), Bernoulli-shift rules, complex Bernoulli-shift rules and hyper-Bernoulli-shift rules. By using random bit strings as testing signals, 30 topologically-distinct Bernoulli-shift rules have been found, via extensive computer simulations and analytical studies, to exhibit robust Bernoulli-shift steady-state behavior. Even more remarkable is their robustness in the sense that any random initial bit string could converge to one of these robust Bernoulli-shift attractors. In particular, each Bernoulli attractor is parameterized by three integers (σ, τ, β) satisfying $\beta = \pm 2^{\sigma}$. A Bernoulli-shift attractor with $\beta > 0$ simply implements a left shift (if $\sigma > 0$) or a right shift (if $\sigma < 0$) of $|\sigma|$ bits in every τ iterations. If $\beta < 0$, the same operation is followed by complementation (i.e. changing symbol of all bits) during each operation (i.e. every τ iteration).

Due to the fact that many topological properties such as topological entropy, sensitivity and topologically mixing of CA are undecidable [12, 13], one should, in principle, separately analyze time-asymptotic dynamics for each Bernoulli-shift rule in the bi-infinite binary sequence space S^Z. At present, their long-term dynamical behaviors are analyzed from the viewpoint of symbolic dynamics [15-20].

2. Complex symbolic dynamics of Bernoulli-shift rules

For a finite symbol set $S = \{0, 1, \cdots, k-1\}$, a word over S is a finite sequence $a = (a_0, \cdots, a_{n-1})$ of elements in S. Denote by $S^* = \bigcup_{n\geq 0} S^n$ the set of words over S, where S^n is the set of all words of length n. If a is a finite or infinite-length word and $I = [i, j]$ is an interval of integers on which a is defined, then write $a_{[i,j]} = (a_i, \cdots, a_j)$ and $a_{[i,j)} = (a_i, \cdots, a_{j-1})$. Also, b is a subword of a, denoted by $b \prec a$, if $b = a_I$ for some interval $I \subseteq Z$; otherwise, denoted by $b \nprec a$. The set of bi-infinite words is denoted by S^Z. In S^Z, the cylinder set of a word $a \in S^n$ is $[a]_k = \{x \in S^Z \mid x_{[k,k+n)} = a\}$, where $k \in Z$. Obviously, the cylinder sets generate the topology on S^Z and form a countable basis for this topology. Every open set is a countable union of some cylinder sets. For a self-map f on S^Z, a set

$X \subseteq S^Z$ is f-invariant if $f(X) \subseteq X$, and strongly f-invariant if $f(X) = X$. If X is closed and f-invariant, then (X, f) or simply X is called a subsystem of (S^Z, f). For instance, let $\mathscr{A}$ denote a set of some finite words over S, and $\Lambda = \Lambda_{\mathscr{A}}$ be the set consisting of the bi-infinite words in $\mathscr{A}$. Then, $\Lambda_{\mathscr{A}}$ is a subsystem of (S^Z, σ), where $\mathscr{A}$ is said to be the determinative block system of Λ with $\sigma : S^Z \longrightarrow S^Z$ defined by $[\sigma(x)]_i = x_{i+1}$. For a closed σ-invariant subset $\Lambda \subseteq S^Z$, the subsystem (Λ, σ) or simply Λ is called a subshift of σ. The topological dynamics of a subshift of finite type is largely determined by the properties of its transition matrix A [14]. A matrix A is positive if all of its entries are non-negative; irreducible if $\forall i, j$, there exists n such that $A^n_{ij} > 0$; aperiodic if there exists M, such that $A^n_{ij} > 0$, $\forall n > M$ and $\forall i, j$. If Λ_A is a 2-order subshift of finite type, then it is topologically mixing if and only if A is aperiodic, where A is its associated transition matrix with $A_{ij} = 1$, if $(i, j) \prec \Lambda$; otherwise $A_{ij} = 0$.

By a theorem of Hedlund [3], a map $f : S^Z \longrightarrow S^Z$ is a cellular automaton if and only if it is continuous and commutes with the shift map σ, i.e., $\sigma \circ f = f \circ \sigma$. For any CA, there exists a radius $r \geq 0$ and a local rule $N : S^{2r+1} \longrightarrow S$ such that $[f(x)]_i = N(x_{[i-r,i+r]})$. Moreover, (S^Z, f) is a compact dynamical system. To enhance readability, it is desirable to write a CA as a global map f_N for local rule N, where N is the decimal notation in light of Wolfram's scheme. For one-dimensional elementary cellular automata (ECA), $r = 1$ and $S = \{0, 1\}$. For example, rule 2 is the one for which $N(0, 0, 1) = 1$ and $N(x_{[i-1,i+1]}) = 0$ for all other triples. Based exclusively on the rule, the following proposition is a direct consequence.

Proposition 1 *For rule 2, there exists a subset* $\Lambda \subset S^Z$ *such that* $f_2|_\Lambda = \sigma|_\Lambda$, *where* $\Lambda = \{x \in S^Z | \ x_{[i-1,i+1]} \in \mathscr{A}, \forall i \in Z\}$ *and* $\mathscr{A} = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$.

According to Proposition 1, dynamical behaviors of f_2 on Λ can be characterized via a subshift Λ, which is a subshift of finite type. Observe that the transition matrix A of the 2-order subshift, which is topologically conjugate to Λ, is

$$A = \begin{pmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \\ 1\,1\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix}.$$

One can easily verify that A is irreducible and aperiodic.

Proposition 2
(**a**) f_2 *is topologically mixing on* Λ*;*
(**b**) Λ *is the nonwandering set* $\Omega(f_2)$ *and the global attractor of* f_2*;*
(**c**) *the topological entropy of* f_2 *equals* $\log \rho(A) = \log \lambda_0 \approx 0.382$, *where* $\rho(A)$ *is the spectral radius of the transition matrix A and* λ_0 *is the positive real root of* $\lambda^3 - \lambda^2 - 1 = 0$.

It is well known that positive topological entropy implies chaos in the sense of Li-Yorke, and topologically mixing is also an indication of complexity of dynamical systems. A system with topologically mixing property has many chaotic properties by different means. In conclusion, the above two propositions have led to the following theorem.

Theorem 1
(**a**) f_2 *is chaotic in the sense of both Li-Yorke and Devaney on the global attractor* Λ*;*
(**b**) f_2 *is chaotic in the sense of Li-Yorke on* S^Z.

Remark 1 *Rules 2, 10, 34, 42, 46, 130, 138 and 162 have the same shifting mode: shift the bit strings to the left by one bit in every iteration, i.e., they have the same Bernoulli-shift attractor parameters* $(\sigma, \tau, \beta) = (1, 1, +)$. *In order to describe the Bernoulli shifting dynamics in an unambiguous way, analogous results for these rules are listed in Table 1. Column* 3 *displays the approximation of topological entropy* $\log \lambda_0$, *where* λ_0 *is the positive real root of the characteristic equation of the corresponding transition matrix. Column* 5 *shows the topologically conjugate rules associated with rule N.*

N	topologically mixing	topological entropy	chaotic property	equivalent rules
2	yes	0.382	Li-Yorke and Devaney	16,192,247
10	yes	0.481	Li-Yorke and Devaney	80,175,245
34	yes	0.481	Li-Yorke and Devaney	48,187,243
42	yes	0.609	Li-Yorke and Devaney	112,171,241
46	yes	0.382	Li-Yorke and Devaney	116,139,209
130	no	0.382	Li-Yorke	144,190,246
138	yes	0.562	Li-Yorke and Devaney	208,174,244
162	no	0.481	Li-Yorke	176,186,242

Table 1: Summary of quantitative properties of robust Bernoulli-shift rules 2, 10, 34, 42, 46, 130, 138 and 162 with the same parameters $(\sigma, \tau, \beta) = (1, 1, +)$.

Similar analytic arguments can also be applied to all other robust Bernoulli-shift rules. In addition to the 8 Bernoulli-shift rules listed so far (with the same Bernoulli parameters), there are 22 additional ones possessing up to three robust Bernoulli attractors. Space limitation precludes a more detailed analysis of the others. Main dynamical properties of typical Bernoulli-shift rules 3, 7, 11, 24, 35, 142, and 74 have been derived and exhibited in Table 2, but for others it is referred to [15-19].

Remark 2 *It has been shown that some special examples of Chua's robust period rules also exhibit non-robust Bernoulli-shift behaviors for both finite and bi-infinite cases [20].*

N	Bernoulli parameters (σ, τ, β)	topolo -gically mixing	topological entropy	equivalent rules
3	$(1, 2, +)$	yes	0.281	17,63,119
7	$(-1, 2, +)$	yes	0.425	21,31,87
11	$(-1, 1, -)$ $(1, 1, +)$	yes yes	0.481 0.199	47,81,117
24	$(-1, 1, +)$	yes	0.382	66,189,231
35	$(1, 1, +)$ $(-1, 2, +)$	yes	0.281 0.562	49,59,115
142	$(1, 1, +)$ $(-1, 1, -)$	yes yes	0.481 0.481	212
74	$(1, 1, +)$ $(2, 2, +)$ $(-3, 3, +)$	yes yes no	0.382 0.464 0.322	88,173,229

Table 2: Summary of quantitative properties of robust Bernoulli-shift rules 7, 24, 35, 142 and 74 with distinct Bernoulli shifting types.

3. Gliders and collisions in Bernoulli-shift rules

ECA rules 54 and 110, which are classical examples of Wolfram's class IV, have been an object of special attention due to their ability to generate rich varieties of periodic structures, known as particles or gliders [7, 21, 22]. A glider is a compact group of non-quiescent states travelling along cellular automata lattice. Gliders are believed to be the characteristic signatures allowing one to recognize class IV behavior, whereas all types of Bernoulli-shift dynamics can also be considered as gliders. Certainly, the catalog of all possible gliders and glider collisions is less plentiful than that of rules 54 and 110. These ideas are formalized under the framework of symbolic dynamics below.

For a CA rule N and its global map f_N, an infinite cyclic configuration x is $x = a^* = (\cdots, a, a, a, \cdots) \in S^Z$, where $a = (a_0, a_1, \cdots, a_{n-1}) \in S^n$. Further, $a_L^* = (\cdots, a, a)$ and $a_R^* = (a, a, \cdots)$ stand for left-infinite cyclic configuration and right-infinite cyclic configuration, respectively. The evolution orbit $\{x, f_N(x), f_N^2(x), f_N^3(x)\cdots\}$, denoted by $Orb_N(x)$, is said to be a background ether or background pattern of rule N, if there exist nonnegative integers m, m', k and k' such that $f_N^m(x) = \sigma_L^k(x)$ and $f_N^{m'}(x) = \sigma_R^{k'}(x)$, where the sequence a is called an ether factor.

Suppose that $x = a^*$ is an infinite cyclic configuration with $a \in S^n$ and $b \in S^{n'}$ such that $b \neq (\overbrace{a, a, \cdots, a}^{k})$, $k \in Z^+$, and let $\tilde{x} = (a_L^*, b, a_R^*)$. Then:

(1) if there exist nonnegative integers m and k such that $f_N^m(\tilde{x}) = \sigma_R^k(\tilde{x})$, then the orbit $Orb_N(\tilde{x})$ is called a right glider of rule N with velocity k/m;

(2) if there exist nonnegative integers m and k such that $f_N^m(\tilde{x}) = \sigma_L^k(\tilde{x})$, then the orbit $Orb_N(\tilde{x})$ is called a left glider of rule N with velocity $-k/m$;

(3) if there exists nonnegative integer m such that $f_N^m(\tilde{x}) = \tilde{x}$, then the orbit $Orb_N(\tilde{x})$ is called a fixation glider of rule N with zero velocity;

(4) if the orbit $Orb_N(\tilde{x})$ is a right (left, fixation) glider, then the subsequence $b \prec \tilde{x}$ is called a glider factor of rule N;

(5) if b and c are two glider factors of rule N with disagreed velocities, then the evolutionary orbit $Orb_N(y)$ is called a glider collision of rule N, where $y = (a_L^*, b, \overbrace{a, \cdots, a}^{k}, c, a_R^*)$.

In the following, the background ether, gliders and glider collisions of rule 35 are further discussed.

Proposition 3 *For ECA rule 35, the evolutionary orbit $Orb_{35}(x)$ is a background ether, where $x = (0, 0, 1)^*$; that is, $(0, 0, 1)$ is an ether factor of rule 35.*

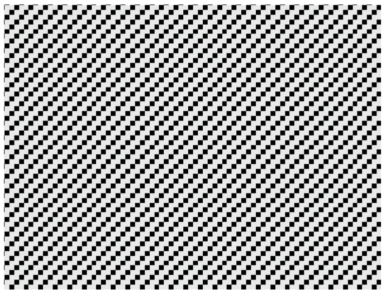

Fig. 1: Background ether of rule 35.

Proposition 4 *Let $y = (a_L^*, b, a_R^*)$, $a = (0, 0, 1)$ and $b = (0, 1)$. Then, the evolutionary orbit $Orb_{35}(y)$ is a left-glider with velocity -1 of rule 35.*

Proposition 5 *Let $y = (a_L^*, b, a_R^*)$, $a = (0, 0, 1)$, and $b = (0)$, $(1, 1)$, $(0, 0, 0)$, $(1, 1, 0)$, $(1, 1, 0, 0, 0)$ and $(1, 1, 0, 0, 0, 0)$), respectively. Then, these evolutionary orbits $Orb_{35}(y)$ are all right-gliders with velocity $1/2$ of rule 35.*

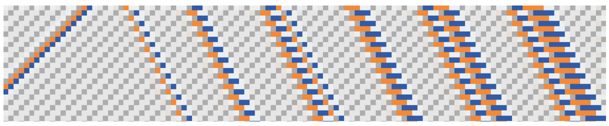

Fig. 2: A left-glider and six right-gliders of rule 35.

Proposition 6 *For $y = (a_L^*, b, \overbrace{a, a, \cdots, a}^{k}, c, a_R^*)$, the evolutionary orbit $Orb_{35}(y)$ is a glider collision of rule 35, if $a = (0, 0, 1)$, $b = (0)$, $c = (0, 1)$, or $a = (0, 0, 1)$, $b = (1, 1, 0)$, $c = (0, 1)$.*

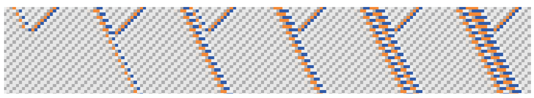

Fig. 3: Glider collisions in rule 35.

It is now clear that there are two types of collision outcomes in rule 35: (1) annihilation of gliders and stationary localization; (2) collision between two gliders leading to another type of glider. For ease of visualization, the background ether, gliders and glider collisions of rule 35 are illustrated in Figs. 1, 2 and 3, respectively, where the black square or blue square stands for "1", and the white square or orange square stands for "0" in the spatiotemporal evolution patterns (colors online).

4. Conclusion

The past few years have witnessed some dramatic advances in the field of cellular automata. One of the main challenges is to explore the quantitative dynamics of these dynamical systems. By taking advantage of robust Bernoulli-shift rules uncovered by Chua and his colleagues, we have developed an elementary yet rigorous proof to explain their chaotic dynamics in view of symbolic dynamics. That is, the intrinsic complexity of all of these Bernoulli-shift rules is quite high according to the usual features that quantify the complexity of discrete dynamics, such as topological entropy and topologically mixing. It is worth mentioning that the qualitative property of glider collision plays an important role in giving a quantitative interpretation of the global dynamics of some CA rules in the bi-infinite sequence space S^Z.

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