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Isle of Eden in 1D binary cellular automaton as a manifestation of Gödel incompleteness and a proposal for a bridge between analytical results and spatial-temporal logic patterns

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Abstract– The richness of spatial-temporal dynamics is known since the Morphogenesis paper of Turing [1] and the introduction of the Cellular Automaton by Von Neumann and Ulam [2]. The recent book of S. Wolfram [3] showcases, by a wealth of examples and one theorem (rule 110), the richness of the simplest spatial temporal binary patterns in the one dimensional (1D) binary cellular automaton, a special case of standard CNN Dynamics [8]. The rigorous study of this model by L. O. Chua and others [4] led to a surprisingly simple and deep insight in the qualitative behavior of these models. The existence of the so called Isle of Eden, a sequence of states without predecessors and successors, is one interesting phenomenon.

The aim of this paper is twofold. (i) We show that the Isle of Eden is a simple manifestation of Gödel's incompleteness theorem by using the way of the original proof of Gödel. (ii) We propose a way to generate results on binary spatial-temporal logic patterns via analytical proofs using the binary to continuous transformation introduced by L. O. Chua et. al. [4].

We note that the 1D Cellular Automaton is a simple case of a CNN dynamics programmably embedded in a CNN Universal Machine [5] and it could be simply implemented on existing cellular camera computers [6], as well as on different cellular many-core chips like FPGAs.

1. Introduction

A brief history from Boole to Gödel shows how logicians starting from the formal logic algebra, the Boolean algebra, developed a massive theory of logic reasoning and this process culminated in the astonishing proof of Gödel's theorem [7]. The latter is showing that in each formal system, defined by a set of axioms and logic operators, there exists at least one statement that is undecidable.

2 The 1D CNN and the binary Cellular Automaton [8]

Standard Cellular Nonlinear/neural Networks (CNNs) are defined on a 1D, 2D or 3D topographic grid, placing mainly identical 1st order nonlinear dynamic systems called cells on the grid points, and introducing mainly local interactions between the cells called cloning templates. If the inputs are binary and the templates are completely stable then the outputs are binary as well. Hence, the 1D Cellular Automaton is a special case of a standard CNN, the sequence of outputs being the input and initial state of the next step in the sequence.

3. The 1D Binary Cellular Automaton generating sequences of binary states (SBS) as Gödel's recursive universe

Gödel's metamathematical method is based on attaching numbers to the signs and series of signs existing in his formal system P via Gödel numbering [7]. It is a one dimensional coordinate metamathematics. There exists a one-to-one correspondence between a sign/string and a natural number $g \in G$ (the set of Gödel numbers) G is a subclass of Z (integers).

Assign natural numbers starting with 2, like 2, 3, 4, 5, etc., to signs, like n1, n2, n3, nk, for k different signs. A series of k signs is represented by an integer $g = 2^{n1} * 3^{n2} * 5^{n3} * 7^{n4} * pk^{nk}$, pk being the k-th prime number starting from 2.

Hence for every class of strings of symbols and signs there is a one-to-one correspondence of a Gödel number.

To any relation ***R*** between strings there is a one-to-one correspondence ***R'*** between Gödel numbers.

For example the metamathematical statement that "*the series s of formulae is a "proof" of formula f*" <u>is true if and only i</u>f a certain arithmetical relation holds between the Gödel numbers of s and f, (**s** and **f**) being the proof of.

Notation: x B y: x is a proof (Beweis) of formula y

Recursiveness: We are applying a series of operators, relations, all together n-times, the n-th one is denoted by R(n) is a recursive definition.

A *class sign α; v* is a series of signs which is a formula which contains exactly one free variable v. A class-sign is recursive if it can be interpreted as a recursive arithmetical class and if v is substituted the number sign will be provable or disprovable. A *recursive relation sign* may contain several free variables.

Proposition VI of Gödel is more general than the existence of undecidable formulae in the formal System P since it contains deductions (not only proofs) from formulae not included among the axioms.

Proposition VI: If the formal System P satisfies certain conditions of consistency, then there exists at least one recursive class sign r in P such that neither v Gen r nor Neg (v Gen r) is provable within P, where v Gen r is the generalization of r with respect to its free variable v.

Now, we are in a position *to represent the 1D binary CA via Gödel's framework.*

We are considering a 1D, 1 neighbor, length L, binary state Cellular Automaton with periodic boundary condition, denoted by 1D1nLbCA or abbreviated as standard 1DCA.

In the table below, we assign the different notions of 1DCA to different notions of a formal system P.

1 D CA	***Formal System P***
Signs: states x_i (2,3,4...N) $x:[b_0, b_1 .....b_{L-1}]$ $N= 2^L$ initial state: x_0 state sequences $S^k \in S^*$ S^k : k states, S^* : trajectory set $x_0, x_1, ... x_{k-1}$ a series of k signs, represented by a Gödel number	Basic signs $\rightarrow g \in G \subset \mathbf{N}$ Free variable v k signes $\rightarrow$ $n_1, n_2 .... n_k$ 2, 3,........k+1 series of k signes of a string s $p_1^{n_1} \cdot p_2^{n_2} \dots p_k^{n_k}$ $2^2.3^3.5^4.7^5.p_k^{k+1}$ p_k: prime numbers
One class of trajectories	One class of G. numbers
CA Rule R 256 different rules Trajectory: applying rule R <u>recursively</u>, initial state x_0	Relation R Recursive definition Recursive class sign with one free variable v
Isle of Eden can not be reached from outside of it, neither another state outside of it can be reached from Isle of Eden (Reached: via recursive rules)	v Gen R can not recursively defined starting from axioms

This means that the Isle of Eden is a simple manifestation in a spatial-temporal setting of the incompleteness.

4. A method to generate results on binary spatial-temporal logic patterns via analytical proofs using the binary to continuous transformation.

The framework [4] analyzing the binary 1D CA contains a very important one- to-one transformation from the spatial binary representation to one continuous value representation of single state. We denote it by BtoC (Binary to Continuous). This map is a bijection, hence the BtoCtoB map is unique in both directions.

The following way would lead to discover new spatial-temporal logic patterns via analytic techniques.

Logic representation of binary patterns ► binary state and dynamics ► BC ► continuous state and dynamics ► theorem on continuous dynamics ► CB ► theorem on spatial binary dynamics ►theorem on logic representation

In this way, a theorem proved rigorously in the real valued domain could be converted back to the logic pattern domain. An example is the proof of fractal properties [10].

References

[1] A. Turing, "The chemical basis of Morphogenesis," *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, vol.237, No. 641. (Aug. 14, 1952), pp. 37-72.1952

[2] J. v. Neumann, "Theory of self-reproducing automata (ed. A. W. Burks)," *University of Illinois Press, Urbana-London*, 1966

[3] S. Wolfram, "A New Kind of Science," *Wolfram Media, Inc. Champaing, IL*, 2002

[4] L.O.Chua et. al. "A Nonlinear Dynamics Perspective of Wolfram's New Kind of Science," World Scientific, Vol. I, 2005.

[5] T.Roska and L.O.Chua, "The CNN Universal Machine-an Analogic Array Computer," *IEEE Trans. on Circuits and Systems II*, vol. 40, pp. 163-173 March 1993

[6] A. Rodriguez Vázquez et al. Ace 16 k.

[7] K. Gödel, "On formally undecidable propositions of Principia Mathematica and related systems, Introduction by R.B.Braithwaite," *Dover Publications, New York*, 1962

[8]. L.O.Chua and T. Roska, "Cellular Neural Networks and Visual Computing," *Cambridge University Press, New York*, 2002 (paperback edition, 2005).

[9]. J. Van Heijenoort, "From Frege to Gödel," *Harvard University Press Cambridge, Massachusetts*, 1970.

[10] L.O.Chua et. al. "A Nonlinear Dynamics Perspective of Wolfram's New Kind of Science. Part IX: Quasi-ergodicity," International Journal of Bifurcation and Chaos, 18(9) pp. 2487-2642, 2008.