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Standard complex network measures of recurrence-based phase space networks constructed from time series

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Abstract—Recently, intensive efforts have been made to transform time series into networks since the networks constructed from time series can provide complementary information to that of the phase space of different dynamical systems. The recurrence-based phase space network, which is built by linking k nearest neighbors of every point in the reconstructed phase space, can be used to specify different types of dynamics in terms of the motif ranking. Unlike network size with different scaling exponents and the degree distribution mimics the behavior of a discrete Gaussian distribution with different bandwidths from periodic to chaotic Rössler systems. These results indicate that network statistics may offer deeper understanding about the organization of the points in phase space. Finally, local network properties such as the vertex degree, the clustering coefficients and betweenness centrality are found to be sensitive to local stabilities of the obits and contain complementary information.

1. Introduction

Recently, a framework for analyzing time series by constructing associated complex networks has attracted research interests [1–7]. Representing the time series through a corresponding network provides new methods to explore the properties of time series based on complex network theory which may lead to deep insights into the dynamical processes of a system. Many methods for transforming time series into networks have been suggested. A natural way for building networks from time series is by adding links between segments with some similarities, based on which, recurrence networks [3], cycle networks [2], correlation networks [5, 6] have been proposed. The visibility graph suggested by Lacasa *et al.* [4] which maps each points of the time series to nodes and link two nodes if a partial convexity constraint is fulfilled, is one of notable alternative methods.

Many research efforts have been devoted to recurrence networks based on the concept of recurrences in the phase space. Traditionally, to construct a recurrence network following the idea of a recurrence plot by Eckmann *et al.* [8], the time series is mapping into a set of points in the embedded phase space. Each point represents a node of the network, and two nodes are linked together when their phase space distance are smaller than a selected threshold ε. By choosing an appropriate ε, the local phase space properties can be best preserved [7]. In 2008, Xu *et al.* suggested constructing networks linking each node with its k nearest neighbors [1] in the reconstructed phase space. At the microscopic level, the building blocks of the complex networks such as the network motifs reflect different local structure properties [9]. Based on the occurrence frequencies of the motif patterns, a so-called superfamily phenomenon is observed which can distinguish time series with different dynamics.

In this paper, we quantify these recurrence-based phase space networks (using the k neighbor method of [1]) via the standard topological statistics besides the motif ranking. Such global and local statistics can provide new information about the phase space geometry within time series.

2. Quantitative assessment of recurrence-based phase space networks

When transforming time series into a recurrence-based phase space network representation, one node in the phase space is connected to its k nearest neighbors and the associated network inherits structure of the distinct local phase space properties from different dynamical systems. The key question is what information is contained inside the network representation. When characterizing the structure of a complex network, we intend to look at its global properties such as the average path length, the global clustering coefficient and the degree distribution. Starting from a low periodic time series, the associated network is regular as the points arrange orderly round an orbit in the phase space. The network forms a big loop shape and its size increases proportionately to the number of points of the embedded data. If we add noise to the data, the dimension of the dynamics increases while the homogenous distribution of the points remain unchanged. we still get the loop structure but the diameter of the network decreases For chaotic time series, the reconstructed phase space becomes heterogenous and may have some fractal properties and thus the associating network generates heterogeneity.

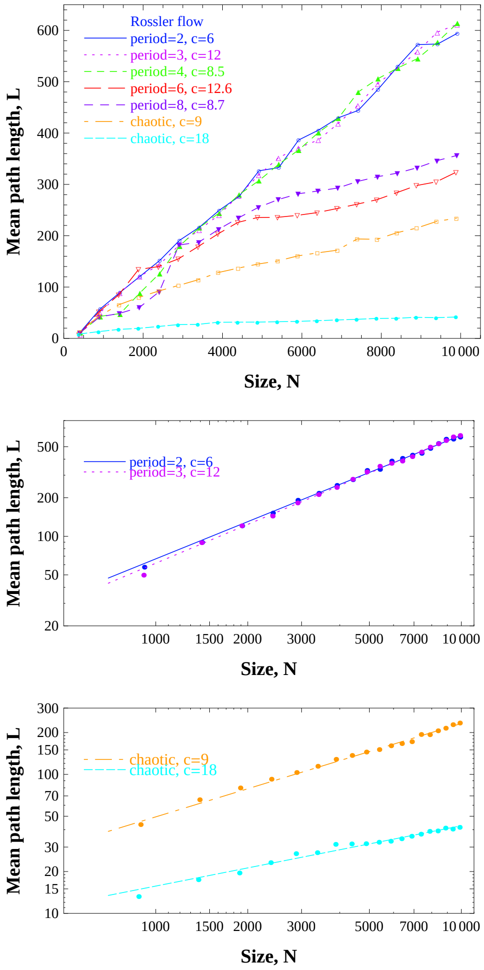

Figure 1: (Upper) Dependence of the average path length $L(N)$ on the length N of the time series for Rössler system of different dynamical types. (Middle) Best fitting models are $L(N) = 0.085N^{0.96}$ and $L(N) = 0.059N^{1.00}$ for low periodic Rössler of period 2 and period 3 respectively. (Lower) Best fitting models are $L(N) = 0.48N^{0.67}$ and $L(N) = 0.78N^{0.43}$ for chaotic Rössler with $c = 9$ and $c = 18$ respectively.

2.1. Global network properties

To investigate the distinction among different dynamical types in more detail, we study the average path length as a function of the time series length. We make use of the data from the x component of the Rössler system $x' = -(y + z)$, $y' = x + ay$, $z' = b + (x - c)z$ with $a = 0.1$ and $b = 0.1$. By tuning the parameter c, low periodic, high periodic and chaotic data can be obtained. The data is mapping into a space of dimension $d_e = 10$ with the time delay τ which is chosen to be the first minimum of the mutual information. Each point in the embedded space represents a node and we link it with its 4 nearest neighborhoods to form the recurrence-based phase space network with the mean degrees equal to 8. Thus, the number of nodes of the network equals to $N - \tau \cdot d_e$ where N is the time series length.

Figure 1 shows how the average path length of the phase space network change with respect to the increasing length of the time series. Note that the average path length can be affected by the choice of initial point of the flow data, so the desired results are taken as an average over several networks from different segments of the same parameter. We find average path length $L(N)$ scales with N with different scaling exponents for low periodic and chaotic Rössler systems. The average path length $L(N)$ increases linearly with time series lengths N for low periodic time series with periods 2 and 3, but exponentially for chaotic Rössler. For high periodic cases, we find transient behavior when the size N is small because the corresponding time series can be too short to cover enough periodic information. Moreover, the slopes for average path length $L(N)$ are larger than the two chaotic Rössler cases but smaller than the low periodic cases due to the interactions between two nearby orbits appear in the phase space.

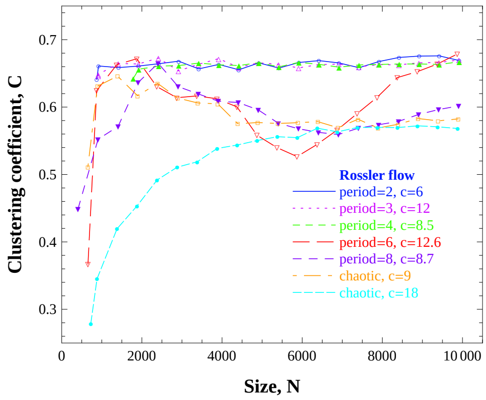

Figure 2: Dependence of the global clustering coefficient $C(N)$ on the length N of the time series.

We also study other global properties including the clustering coefficient and the degree distribution for Rössler systems of different dynamical type as shown in Figs. 2 and 3. The global clustering is closely related to motif ranking. As motif F that denotes a fully-connected subgraph of order 4 indicates strong mutual coupling between nodes, we may expect that the clustering coefficient for the periodic Rössler will be larger than the chaotic Rössler, because motif F occurs more frequently in periodic phase space network rather than network from chaotic data of the same Rössler system as mentioned in Ref. [1]. From Fig. 2 we can see that the clustering coefficients are rather stable with respect to an increasing N and converges to around 0.68 for low periodic with period = 2, 3 as network size increases. Similar behavior can be found in the chaotic Rössler with $c = 18$. The clustering coefficients are also stable but the curve converges to a lower value around 0.58. In comparison, significant transient effect can be found in the remaining three cases. The curve for the chaotic Rössler with $c = 9$ first rises to a same high level as the low periodic cases for the N between 1000 and 2000. Then, it turns down and finally converges to a low level as the chaotic case. The curve for Rössler of period 6 goes up and down and finally goes to the same value as low periodic cases. Although the value of clustering coefficient for Rössler of period 8 is closer to that of the chaotic cases, we may infer that, as the size of the network increases, the curve will finally come up to the low periodic values.

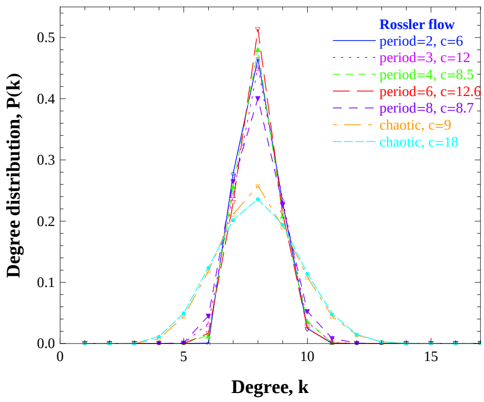

Figure 3: The degree distribution for the Rössler system of different dynamical types.

Figure 3 shows the average degree distributions for networks of several realizations associated with 10000 points of the Rössler data with different dynamics. We find that the average degree distribution mimics the behavior of discrete Gaussian distribution for networks from the Rössler system in different dynamical regimes. The degree at which $p(k)$ reaches its maximum is 8, which is the average degree of the network. The peak values of $p(k)$ for networks generated from periodic data are all above 0.3 and the bandwidths of the Gaussian distribution are rather narrow, indicating that the networks are homogenous. In contrast, for chaotic Rössler, $p(k)$ will be lower than 0.3 and the bandwidths of the Gaussian distributions are wider.

2.2. Local network properties

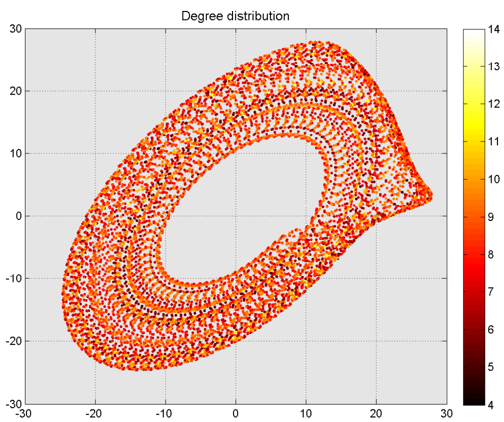

Figure 4: Color coded representation of vertex degree on the attractor for chaotic Rössler with $c = 18$.

Rather than the global properties, we also examine local vertex properties which can provide more detailed information which is otherwise buried in the geometry of the attractor. The first global property we study is the local vertex degree which is the number of neighbors of a given vertex v in the network. The degree of a vertex for the traditional recurrence networks which based on an appropriate choice ε reflects exactly the local recurrence rate. When it comes to phase space network, we should recall that the average degrees are 8 due to the distinctive construction methods we use to obtain the network. Thus the local degrees depend closely on the spatial filling of the phase space attractor. On the one hand, in the region where the density of the phase space is homogenous, the corresponding structures of the network retain the homogeneity from the attractor and local degrees for the vertices will be around 8. That is, every point in these regions is connected to its 4 nearest neighbors and at the same time gains around 4 extra links from points which take it as a nearest neighbor. One the other hand, if points locate in the regions where there are significant changes in the phase space density, some of these points may gain more than 4 extra links besides its 4 nominal nearest neighbors, resulting in the lack of neighbors for some other points in the same region. These regions will have a heterogeneous degree distribution. Long periodic signals follow a uniform distribution of low dimension while the trajectories of chaotic signals tend to be trapped in the vicinity of the unstable periodic obits that lead to density changes in the phase space. One may refer to color coded representation of vertex degree on the attractor for chaotic Rössler with $c = 18$ in Fig. 4 for details. Also, we can address the different bandwidths in Fig. 3 which shows the degree distribution for the Rössler of different dynamical types. Since more heterogeneity is generated by the phase space of chaotic Rössler, the bandwidths will be wider.

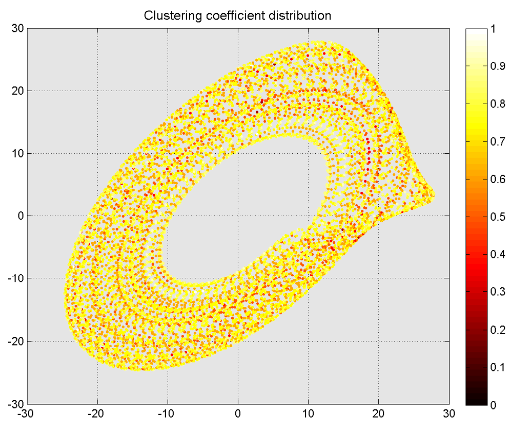

Figure 5: Color coded representation of vertex clustering coefficient on the attractor for chaotic Rössler with $c = 18$.

The second property we study is the clustering coefficient of a specific vertex which quantifies the relative density of connections between its neighbors and thus measures the network in a relative larger scale compared to the local degree. Figure 5 shows color representation of vertex clustering coefficient on the attractor for the chaotic Rössler with $c = 18$, from which we observe the heterogeneously distributed local clustering coefficients. As the heterogeneity comes from the heterogenous distribution in the density of the phase space, one might expect that a node in the dense region of the phase space will naturally have a larger clustering coefficient and vice versa. However, it is shown in Fig. 5 that the values of clustering coefficient for some nodes locate in the dense region of the phase space could be small while the values for nodes locate the boundary region of the attractor could be large. In a homogenously filled phase space region, things become simpler and we can see points with relatively uniform and higher clustering coefficients. However, when it comes to a region with large density fluctuations, we have to consider the interactions between several orbits which are close to each other. A strong connectivity between neighbors of the same point could be obtained only if its neighbors tend to lie in different orbits, because we have excluded links within the same orbit when constructing the phase space networks. We can conclude that the local clustering coefficient contains the complexity of the interaction between orbits in the phase space and can provide complementary information of the system.

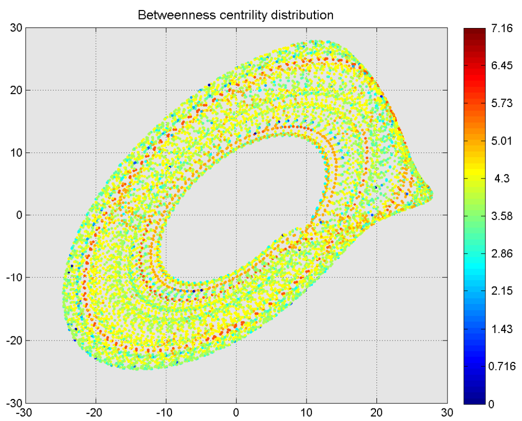

Figure 6: Color coded representation of logarithm of vertex betweenness centrility, $\log(b_v + 1)$,on the attractor for the chaotic Rössler with $c = 18$.

The third property we study is the betweenness centrality of a specific vertex which quantifies the relative shortest path length pass through the vertex and thus it is more complicated because it measures the network in an even larger scale compared to the clustering coefficient and the local degree. What we are most interested in are the points with a large betweenness centrality which are of importance for the many shortest paths on the network. As shown in Fig. 6, the nodes with high betweenness centrality may correspond to the region with low phase space density between such regions because of an increasing number of shortest path between such regions. For vertices with large degree in the dense region of the phase space close to the unstable periodic obits, low values of betweenness centrality can be found. A similar conclusion has been drawn for the traditional recurrence networks [7].

3. Conclusions

In this paper, we apply standard complex network measures to recurrence-based phase space network constructed by linking every point with its k nearest neighbors in reconstructed phase space. In particular, global properties such as average path length, the global clustering coefficients and degree distribution can be used to distinguish system of different dynamical types. Local degrees, clustering coefficients and betweenness centrality reveal information on the spatial fillings of the phase space and are close to dynamical invariants such as the unstable periodic obits. The specific vertex properties can provide more detailed information about the local attractor geometry in the phase space.

Acknowledgments

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