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Linearity and nonlinearity within recurrence plots

Yoshito Hirata and Kazuyuki Aihara

Institute of Industrial Science, The University of Tokyo, Tokyo 153-8505, Japan
Email: yoshito@sat.t.u-tokyo.ac.jp

Abstract—Recurrence plots are widely used for analysing time series generated from complex systems. While it is argued that the tau-recurrence rate is similar to the autocorrelation, it is still not clear which characteristics within recurrence plots represent their linear and nonlinear properties. In this paper, we observe how fast the number of diagonal lines decreases when their length increases. We found that in linear stochastic systems, the logarithm of the number scales exactly, while in typical nonlinear deterministic systems, the decreasing rate of the number tends to approach from the above. We illustrate our finding with numerical examples.

1. Introduction

Nonlinear time series analysis [1, 2, 3] has been developed for the last three decades. Among various methods, recurrence plots [4, 5] have been intensively investigated for the last two decades.

Recurrence plots visualize given time series. Recurrence plots are two-dimensional graphs both axes of which represent time. For a pair of two times, if the corresponding states are similar, then one plots a point at the corresponding place. Otherwise, one plots nothing. Due to the simple definition, we can extend recurrence plots to marked point processes [6]. Recent trends in recurrence plots are their quantifications, which are called Recurrence Quantification Analysis (RQA) [5]. Quantities that characterize vertical and horizontal lines in recurrence plots were proposed. Using these quantities, the properties of given time series can be characterized from various viewpoints.

Although recurrence plots were studied well, it is still not clear which characteristics represent linear and nonlinear features of given time series within a recurrence plot. There is a paper by Zbilut and Marwan [7] that showed the tau-recurrence rate is similar to the autocorrelation, which characterizes linear stochastic systems.

In this paper, we show that the distribution of the diagonal line length of recurrence plots can distinguish linear stochastic systems from nonlinear deterministic systems. In linear stochastic systems, the logarithm of the number of the diagonal line length scales even if the length of the diagonal line is short. On the other hand, in typical nonlinear deterministic systems, the decreasing rate of the number tends to converge from the above.

The rest of the paper is organized in the following way. In Section 2, we introduce recurrence plots and their quantifications. In Section 3, we present numerical examples showing that the distribution of diagonal line length is different between linear stochastic systems and nonlinear deterministic systems. In Section 4, we apply our finding to a dataset of squid giant axon. In Section 5, we conclude the paper.

2. Recurrence plots and their quantifications (RQA)

Let $x_i \in \mathbb{R}^m$ $(i = 1, 2, \cdots, n)$ be a point of a time series at time i. Let $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be a distance function. Here we use the Euclidean distance. A recurrence plot is defined as

$$R(i, j) = \begin{cases} 1, & \text{if } d(x_i, x_j) < \epsilon, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

When $R(i, j) = 1$, a point is plotted at (i, j). If $R(i, j) = 0$, nothing is plotted at (i, j). It is demonstrated that from recurrence plots one can reproduce rough shapes of the original time series [8, 9]. In fact, it was shown that if two recurrence plots are the same, then the corresponding dynamics are equivalent [10]. By using recurrence plots, driving forces affecting observed elements can be reconstructed [14]. Therefore, recurrence plots are good methods for describing and analyzing time series.

Recently recurrence plots are applied to identify network topology from time series [11, 12, 13].

A good point of recurrence plots is that we can understand the characteristics of dynamics from the patterns of the plotted points. For example, from white noise, we can obtain a recurrence plot in which points spread uniformly randomly. From a periodic time series, we can obtain a recurrence plot in which diagonal lines are running in parallel with an equal space. As for a time series generated from a deterministic system, the corresponding recurrence plot contains short diagonal segments of lines. Therefore, diagonal lines characterize the determinism of given time series.

On the other hand, vertical and horizontal lines appear when a point in a time series tends to stay in a particular state in phase space.

There are various quantities characterizing plotted patterns of points in recurrence plots. Analysis calculating these quantities is called Recurrence Quantification Analysis (RQA). The quantities of RQA can be classified into two groups [5]: One of the groups characterizes the distribution of diagonal lines. Let $c(l)$ be the number of diagonal lines with at least length l in a recurrence plot R. This $c(l)$ can be calculated as

$$c(l) = \sum_{i,j=1}^{n} \prod_{k=0}^{l-1} R(i+k, j+k). \quad (2)$$

It is known that from the asymptotic decreasing rate of $c(l)$, one can estimate the correlation entropy [15].

In the next section, we look at the distribution of diagonal line length $c(l)$ closely.

3. Distribution of diagonal line length

We investigate the distribution of diagonal line length for the following four models.

The first model is the first order autoregressive linear model:

$$x(t) = -0.7x(t-1) + \eta_t, \quad (3)$$

where η_t is the Gaussian distribution of the mean 0 and the standard deviation 1.

The second model is the 100th order autoregressive linear model:

$$x(t) = \sum_{i=1}^{100} a_i x(t-i) + \eta_t. \quad (4)$$

We chose a_i from the Gaussian distribution of the mean 0 and the standard deviation 0.05^2 so that the system becomes stable.

The third model is the Ikeda map:

$$\begin{pmatrix} x(t) \\ y(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} 1 + 0.9(x(t-1)\cos(\theta(t)) - y(t-1)\sin(\theta(t))) \\ 0.9(x(t-1)\sin(\theta(t)) + y(t-1)\cos(\theta(t))) \\ 0.4 - \frac{6.0}{1+x(t-1)^2+y(t-1)^2} \end{pmatrix}. \quad (5)$$

We use $x(t)$ as a scalar time series.

The fourth model is the Rössler model:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -(v+w) \\ u + 0.36v \\ 0.4 + w(u-4.5) \end{pmatrix}. \quad (6)$$

We used $u(t)$ sampled every 1 as a scalar time series.

We used the absolute distance for calculating distances. We also chose thresholds so that the recurrence rates became 0.05.

The results are shown in Fig. 1. In the case of linear stochastic systems, the logarithm of the number of the diagonal line length scales almost exactly, while in the nonlinear deterministic systems, the decreasing rate of the number tends to converge from the above.

We showed analytically that in stationary ergodic linear stochastic systems, the logarithm of the number of the diagonal line length scales even if the line length is short. On the other hand, in nonlinear deterministic systems whose dimension is more than 1, due to the effects of false nearest neighbors [16], the deceasing rate of the number tends to converge from the above.

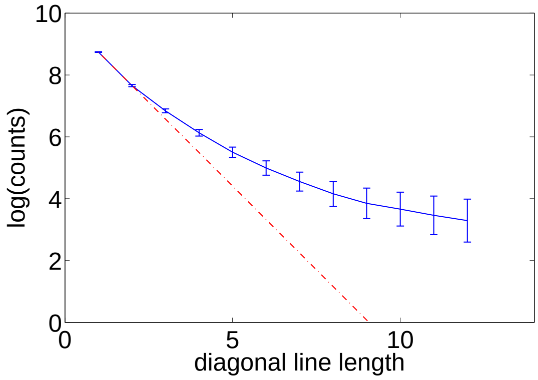

Figure 2: The diagonal line length distribution for the squid giant axon data. The error bars were obtained using the bootstrap method [19]. The broken line was obtained by fitting the logarithms of the counts for lengths 1 and 2.

4. Application to squid giant axon

We applied our finding to the dataset of squid giant axon [17, 18]. This dataset is an ideal example of deterministic chaos since the Lyapunov exponent obtained from various methods [17] matches the metric entropy estimated using symbolic dynamics [18].

We obtained a recurrence plot of the dataset using a threshold such that the recurrence rate is 0.05. The diagonal line length distribution of this dataset is shown in Fig. 2. Since the length of time series is short, we cannot obtained the error bars by using different time series as we did in Fig. 1. Therefore, instead, we obtained the error bars by using bootstrapping the diagonal line length distribution [19]. We can see that the distribution of the diagonal line length is different from that of linear stochastic systems.

5. Conclusion

In this paper, we have shown that the diagonal line length distribution is different from linear stochastic systems with typical nonlinear deterministic systems. In linear stochastic systems, the logarithm of the number of diagonal line length scales even if the line length is short. On the other hand, in nonlinear deterministic systems, the decreasing rate of the number of diagonal line length tends to converge form the above. As for the actual squid giant axon dataset, we found that it is different from linear stochastic systems. The proposed method is simpler than the other existing methods for testing nonlinearity such as ones using time-reversibility [20] and the Fourier transform-based surrogate data [21, 22]. Therefore, we believe that it will contribute to advancements of nonlinear science.

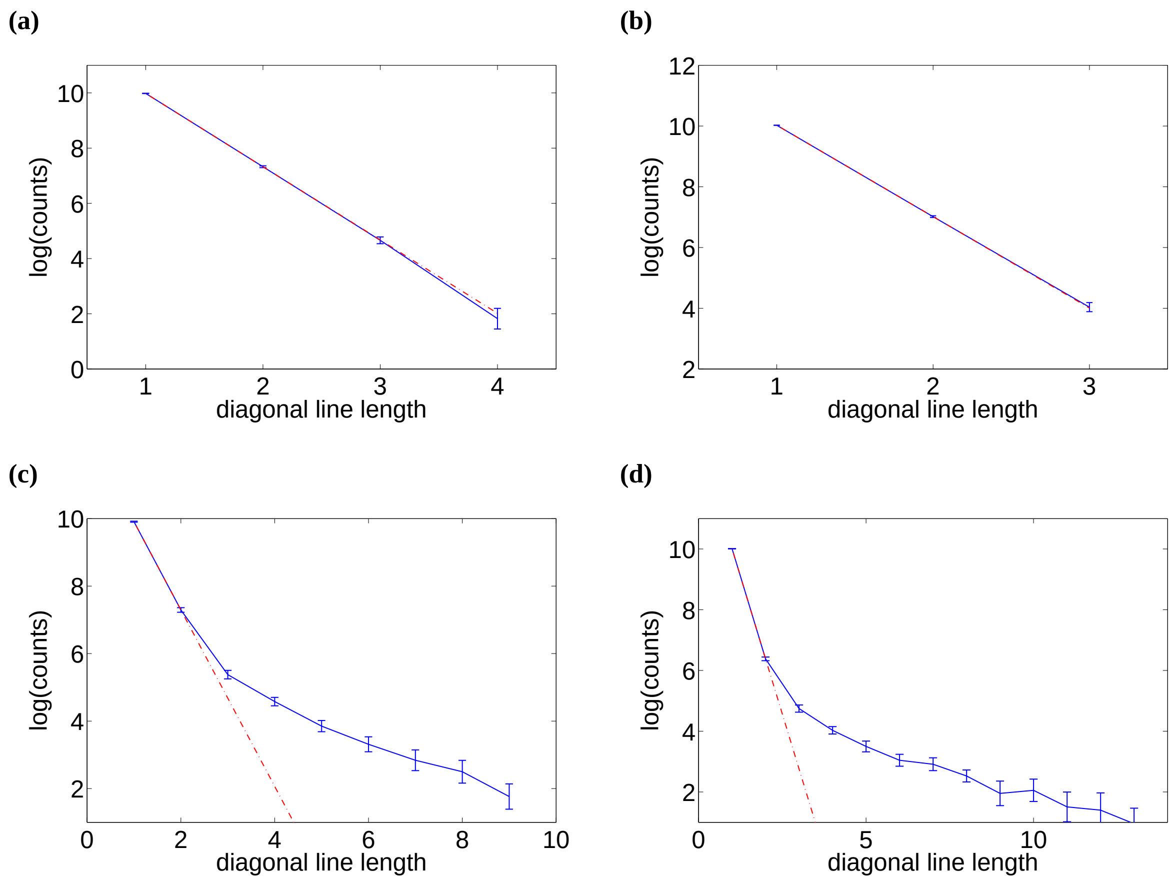

Figure 1: The distribution of number of diagonal line length. (a) the first order autoregressive linear model. (b) the 100th order autoregressive linear model. (c) the Ikeda map. (d) the Rossler model. The error bars were obtained by using 20 different realizations of time series generated from different initial conditions. The broken line was obtained by fitting the means of the logarithms of the numbers for lengths 1 and 2.

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