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Negative β-encoder

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***Abstract*—I.Daubechies et al. have shown that a new class of unconventional analog-digital (A/D) converters, called β-encoder, exhibits exponential accuracy in bit rates while possessing self-correction property for fluctuations of amplifier factor β and quantizer threshold ν. This paper gives optimal values of β and ν to be designed with their unknown errors for given scale, bit budget and tolerance of quantizer in the β-encoder. Futhermore, a new A/D converter, called negative β-encoder, is introduced so as to improve the variance of quantization error of β-encoder.**

I. Introduction

Analog-to-digital (A/D) conversion is the basic technology behind various technologies[1], [2], [3] for example, communications like software radio, audio and imaging, etc. Our main concerns are its accuracy and stability.

The phases of A/D conversion consist of "sampling" and "quantization". Sampling is temporal discretization of signals. The "sampling theorem" guarantees that the signal can be completely reconstructed with sampled values. Quantization is the process of encoding sampled values where quantization error inevitably arises. The problems we focus on are how to reduce the quantization error and guarantee the robustness to the fluctuation of circuit components.

Pulse code modulation (PCM) is known as the A/D converter where the quantization error attains the minimum. Define a quantizer function Q_1 by

$$Q_1(x) := \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases} \tag{1}$$

The bit sequences b_i, $i = 1, 2, \cdots$ can be calculated recursively in the following algorithm. Let $u_1 := 2x$; the first bit b_1 is then given by $b_1 := Q_1(u_1)$. The remaining bits are calculated recursively; if u_i and b_i have been defined, we let $u_{i+1} := 2(u_i - b_i)$ and $b_{i+1} := Q_1(u_{i+1})$.

For $x \in [0, 1)$, bit sequences $\{b_i\}_{i=1}^{L}$, $L \in \mathbb{N}$ are obtained by iterating the Bernoulli shift map $B(x) : [0, 1) \to [0, 1)$ defined by

$$B(x) := 2x \bmod 1 = \begin{cases} 2x, & x \in [0, \frac{1}{2}) \\ 2x - 1, & x \in [\frac{1}{2}, 1) \end{cases} \tag{2}$$

and generating its bit sequence $b_i (i = 1, 2, \cdots)$ defined by

$$b_i := \begin{cases} 0, & B^i(x) \in [0, 1/2) \\ 1, & B^i(x) \in [1/2, 1). \end{cases} \tag{3}$$

Hence we get $B^L(x) = 2^L x - \sum_{i=1}^{L} b_i 2^{L-i}$ which yields

$$x = \sum_{i=1}^{L} b_i 2^{-i} + 2^{-L} B^L(x). \tag{4}$$

When $L \to \infty$, the binary expansion of x has the form

$$x = \sum_{i=1}^{\infty} b_i 2^{-i}. \tag{5}$$

However, if the threshold fluctuates, then the corresponding map $B'(x)$ is defined by

$$B'(x) := \begin{cases} 2x, & x \in [0, \frac{1}{2} + \delta) \\ 2x - 1, & x \in [\frac{1}{2} + \delta, 1), \end{cases} \tag{6}$$

namely, $B'(x) : [0, 1) \to [0, 1 + 2\delta)$ which implies that values of PCM diverge so that PCM doesn't have the robustness to the fluctuation of the quantizer threshold.

The β-expansion as a basis of β-encoder is a classic of ergodic theory [4], [5], [6], [7], [8], [9], [10]. Rényi[4] defined the β-transformation: $x \mapsto \beta x \bmod 1$ for a real number $x \in (0, 1]$ and a real number $\beta > 1$. Gelfond[5] and Parry[6] gave its invariant measure. Parry[7] defined the linear $\bmod 1$ transformation (or (β, α)-transformation, generalized Rényi map): $x \mapsto \beta x + \alpha \bmod 1$ for a real number $x \in (0, 1]$ and real numbers $\beta > 1, 0 < \alpha < 1$ and gave its invariant measure. Dajani[10] discussed the ergodic property of (β, α)-transformation. The λ-expansion[11] has a close relationship to β-expansion.

I.Daubechies et al. [12], [13], [14], [15] introduced an A/D converter having the robustness to the fluctuation of the quantizer threshold ν and the amplication factor $\beta \in (1, 2)$, called β-encoder as shown in Fig 1, where the quantizer Q_ν with its threshold $\nu \in [1, (\beta - 1)^{-1}]$ is defined by

$$Q_\nu(x) := \begin{cases} 0, & x < \nu \\ 1, & x \geq \nu. \end{cases} \tag{7}$$

The β-encoder can perform rightly as far as $\nu \in [1, (\beta - 1)^{-1})$ even if the quantizer threshold ν fluctuates. The bit sequence b_i can be obtained recursively as follows. Let $u_1 := \beta x$; the first bit b_1 is then given by $b_1 := Q_\nu(u_1)$. The remaining bits are given recursively; if u_i and b_i have been defined, we let $u_{i+1} := \beta(u_i - b_i)$ and $b_{i+1} := Q_\nu(u_{i+1})$. So, the β-encoder is regarded as a successful circuit realization of (β, α)-transformation by setting $\nu - 1 = \alpha$[10].

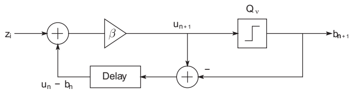

Fig. 1. β-encoder: With input $z = x \in [0, 1)$; $z = 0$ for $i > 0$ and "initial conditions" $u_0 = b_0 = 0$, the output b_{n+1} of this block diagram gives the β-expansion for x by using the quantizer Q_ν with $\nu \in [1, (\beta - 1)^{-1}]$. The cases where $\nu = 1$, $(\beta - 1)^{-1}$ are "greedy" and "lazy" schemes, respectively. The case where $\beta = 2$ and $\nu = 1$ shows the structure of PCM.

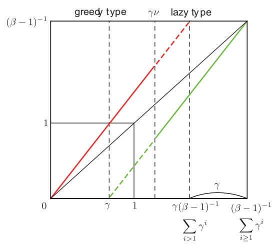

Fig. 2. $N_{\beta,\alpha}$ for $\beta = \sqrt{2}$.

Assume $1 < \beta < 2$. Then each $x \in [0, (\beta - 1)^{-1}]$ has a representation $x = \sum_{i=1}^{\infty} b_i \beta^{-i}, b_i \in \{0, 1\}$. Introducing $\gamma := 1/\beta$, we get

$$x = \sum_{i=1}^{\infty} b_i \gamma^i, \quad b_i \in \{0, 1\}. \tag{8}$$

P. Erdös [8] introduced the lexicographic order $\overset{L}{<}$ between real sequences [9]:$(b_i) \overset{L}{<} (b'_i)$ if there is a positive integer m such that $b_i = b'_i$ for all $i < m$ and $b_m < b'_m$. It is easy to verify that for every fixed $x \in [0, (\beta - 1)^{-1}]$ in the set of all expansions of x there is a greatest and a smallest element with respect to this order: the so-called *greedy* and *lazy* expansion. (The greedy expansions were studied earlier in [4] where they were called β-expansions.) A number x has a unique expansion if and only if its greedy and lazy expansions coincide. K. Dajani [10] introduced the (β, α) expansion, a class of series expansions to $\beta > 1$, $\beta \notin \mathbb{Z}$ for each $\alpha \in [0, \frac{\lfloor \beta \rfloor}{\beta - 1} - 1]$. Let $d_1, \cdots, d_{\lfloor \beta \rfloor}$ be partition points given by: $d_i := \frac{\alpha + 1}{\beta}$, $i = 1, \cdots, \lfloor \beta \rfloor$. Assume $1 < \beta < 2$. Then, the (β, α) expansion map $N_{\beta,\alpha} : [0, (\beta - 1)^{-1}] \to [0, (\beta - 1)^{-1}]$, with its invariant subinterval $[\alpha, \alpha + 1)$, is defined by

$$N_{\beta,\alpha} = \begin{cases} \beta x, & x \in [0, \frac{\alpha+1}{\beta}) \\ \beta x - 1, & x \in [\frac{\alpha+1}{\beta}, \frac{1}{\beta - 1}). \end{cases}$$

When $\alpha = 0$ and $\alpha = (\beta - 1)^{-1} - 1$, the map generates greedy and lazy expansion, respectively as shown in Fig.2.

The digits of the greedy and lazy expansions are defined recursively as follows [9]: If $m \geq 1$ and if the digit b_i of greedy expansion of x is defined for all $i < m$, then

$$b_m = \begin{cases} 1 & \text{if } \sum_{i<m} b_i \gamma^i + \gamma^m \leq x, \\ 0 & \text{if } \sum_{i<m} b_i \gamma^i + \gamma^m > x. \end{cases} \tag{9}$$

If $m \geq 1$ and if the digit b_i of lazy expansion of x is defined for all $i < m$, then

$$b_m = \begin{cases} 0 & \text{if } \sum_{i<m} b_i \gamma^i + \sum_{i>m} \gamma^i \geq x, \\ 1 & \text{if } \sum_{i<m} b_i \gamma^i + \sum_{i>m} \gamma^i < x. \end{cases} \tag{10}$$

The relation $\sum_{i<m} b_i \gamma^i + \sum_{i>m} \gamma^i = \sum_{i=1}^{\infty} \gamma^i - \{\sum_{i<m} \overline{b}_i \gamma^i + \gamma^m\} = (\beta - 1)^{-1} - \{\sum_{i<m} \overline{b}_i \gamma^i + \gamma^m\}$ gives the lazy expansion in a different form of (10), defined by

$$b_m = \begin{cases} 0 & \text{if} (\beta - 1)^{-1} - \sum_{i<m} \overline{b}_i \gamma^i - \gamma^m \geq x, \\ 1 & \text{if} (\beta - 1)^{-1} - \sum_{i<m} \overline{b}_i \gamma^i - \gamma^m < x, \end{cases} \tag{11}$$

where $\overline{b}_i = 1 - b_i$. Comparing (9) with (11), we find that the greedy expansion $\sum_{i<m} b_i \gamma^i$ of x corresponds to the lazy expansion $\sum_{i<m} \overline{b}_i \gamma^i$ of $(\beta - 1)^{-1} - x$. This implies that the greedy expansion is dual to the lazy expansion.

II. Interval Partition by β-expansion

The duality of greedy expansion and lazy expansion suggests that they should be equal, deserving the same performance of decoding process as each other and furthermore, expansions with the quantizer threshold $\nu \sim (1 + (\beta - 1)^{-1})/2$, called "cautious"(neither greedy nor lazy) expansion, will be a good promise other than these two expansions (see Fig.6). To observe remarkable differences between Daubechies' decoded values of x[12], [13][1] and ours[16], discuss the process of interval partition by β-encoding.

Let b_i^β for $i = 1, 2, \cdots, L$ be a β-expansion of x. Since $x = \sum_{i=1}^{L} b_i^\beta \gamma^i + \gamma^L N_{\beta,\alpha}^L(x)$ and $N_{\beta,\alpha}^L(x) \in [0, (\beta - 1)^{-1})$, the interval $I_{L,\beta}(b_i^\beta)$ where x exists is defined by

$$I_L^\beta(b_i) = [\sum_{i=1}^{L} b_i^\beta \gamma^i, \sum_{i=1}^{L} b_i^\beta \gamma^i + (\beta - 1)^{-1} \gamma^L). \tag{12}$$

This means that iterating the β-transform gives the ratio between $(L - 1)$-th successive subinterval-width, defined by $\frac{|I_L^\beta(b_i^\beta)|}{|I_{L-1}^\beta(b_i^\beta)|}$, is equal to γ.

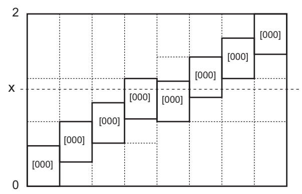

Fig. 3. Interval partitions by β-expansion: x possesses three possible binary representations; 011, 100, 101.

Fig.3 shows that the intervals of uniform width divided by third iterate of the β-transformation overlap and so x has three possible binary representations. Equation (12) leads us to conclude that the middle point of the interval $I_{L,\beta}(b_i)$ [16] should be taken as the decoded value of x, defined by

$$\tilde{x}_L^\beta = \sum_{i=1}^{L} b_i^\beta \gamma^i + (\beta - 1)^{-1} \gamma^L / 2 \tag{13}$$

[1]Dajani[10] also discussed the greedy expansion's superiority over the lazy one.

so that the bound of error between x and $\tilde{x}_\beta$ is given by

$$0 \le |x - \tilde{x}_L^\beta| \le (\beta - 1)^{-1}\gamma^L/2 \tag{14}$$

On the contrary, Daubechies used the leftmost point of the interval $I_{L,\beta}(b_i^\beta)$ as the decoded value of x, defined by

$$\tilde{x}_{L\text{Daub}} = \sum_{i=1}^{L} b_i^\beta \gamma^i \tag{15}$$

and estimated its approximation error given by

$$0 \le x - \tilde{x}_{L\text{Daub}} \le \nu\gamma^L. \tag{16}$$

The bound of eq.(14) is improved by 3 dB than the one of eq.(16) when $\beta > 3/2$. Borrowing Daubechies' idea and using β-expansion sequences b_i^β for x and c_i^β for $y = 1 - x$ $(1 \le i \le L)$, we get the equation for estimating β, called characteristic equation of β [16], defined by (see Fig.7)

$$P_\beta(\gamma) = 1 - \sum_{i=1}^{L}(b_i^\beta + c_i^\beta)\gamma^i - \frac{\gamma^{L+1}}{(1-\gamma)} = 0 \tag{17}$$

which differs from the Daubechies' one [13] given by

$$P_{\text{Daub}}(\gamma) = 1 - \sum_{i=1}^{L}(b_i^\beta + c_i^\beta)\gamma^i = 0. \tag{18}$$

III. β-ENCODER WITH SCALED MAP OF β-EXPANSION

Consider the optimal design of β-encoder with its two parameters $1 < \beta < 2$ and $\nu \in [1, (\beta-1)^{-1})$ in terms of the quantization error against fluctuations of β, ν to be designed. Since the tolerance of threshold fluctuations σ_β is defined by $\sigma_\beta = (\beta-1)^{-1} - 1$, in order to design β and σ_β independently, define the scaled-map with its scale s by

$$S(x) := \begin{cases} \beta x, & x \in [0, \nu\gamma) \\ s - \beta(s - x), & x \in [\nu\gamma, s) \end{cases} \tag{19}$$

where $s > 1$ and $\nu \in [s(\beta-1), s]$. Then, the torelance $\sigma_{\beta,s} = s - s(\beta-1) = s(2-\beta)$. Let $b_i^{S\beta}$ for $1 \le i \le L \in \mathbb{N}$ be a binary expansion of $S(x)$ with using $Q_\nu(\cdot)$ for $x \in [0,1)$, then

$$S^L(x) = \beta^L x - s(\beta-1)\sum_{i=1}^{L} b_i^{S\beta}\beta^{L-i} \tag{20}$$

$$\text{or} \quad x = s(\beta-1)\sum_{i=1}^{L} b_i^{S\beta}\gamma^i + \gamma^L S^L(x). \tag{21}$$

Using $S^L(x) \in [0, s)$, we get its subinterval where x exists

$$I_{L,\beta,s}(b_i^{S\beta}) = [s(\beta-1)\sum_{i=1}^{L} b_i^{S\beta}\gamma^i, s(\beta-1)\sum_{i=1}^{L} b_i^{S\beta}\gamma^i + s\gamma^L). \tag{22}$$

For given bit budget L, scale s and tolerance $\sigma_{\beta,s}$, differentiating $|I_{L,\beta,s}(b_i)|$ with respect to β, i.e.,

$$\frac{d|I_{L,\beta,s}(b_{S\beta i})|}{d\beta} = \frac{\sigma_{\beta,s}\beta^{-L+1}}{(2-\beta)^{-2}}\{\beta - L(2-\beta)\} \tag{23}$$

gives the value of β attaining the minimum of $|I_{L,\beta,s}(b_{S\beta i})|$:

$$\beta_{\text{opt}} = \frac{2L}{L+1}. \tag{24}$$

IV. NEGATIVE β-ENCODING

TABLE I
THE INVARIANT SUBINTERVAL OF NEGATIVE β-EXPANSION

thredhold ν	invariant subinterval
$(\beta-1)s < \nu < \frac{\beta^2-\beta+1}{\beta+1}s$	$[\beta\nu - (\beta^2-\beta)s, \beta s - \nu)$
$\frac{\beta^2-\beta+1}{\beta+1}s \le \nu \le \frac{2\beta-1}{\beta+1}s$	$[s-\nu, \beta s - \nu)$
$\frac{2\beta-1}{\beta+1}s < \nu < s$	$[s-\nu, \beta\nu - (\beta-1)s)$

Define the variance of quantization errors between sampled value x and its decoded value $\tilde{x}$ with sample number N by $V := \frac{1}{N}\sum_{i=1}^{N}|x_i - \tilde{x}_i|^2$. Since the midpoint of the interval $I_{L,\beta,s}$ is chosen as the decoded value, the variance V is larger when $\nu = 1$(greedy expansion) and $\nu = (\beta-1)^{-1}$(lazy expansion) as shown in Fig.6. This fact comes from the result that the invariant subinterval of β-expansion $[\nu, \nu+1)$ is deviated from β-transformation's domain $(0, s]$.

Such a situation motivates us to introduce the *negative β-map* with its scale $s > 1$ defined by

$$R(x) := \begin{cases} s - \beta x, & x \in [0, \nu\gamma) \\ \beta s - \beta x, & x \in [\nu\gamma, s) \end{cases} \quad \nu \in [s(\beta-1), s] \tag{25}$$

as shown in Figure 4.

The invariant subinterval under the map $R(x)$ is a function of ν as shown in Table 1. This, however, shows that each invariant subinterval is always located in the center of $(0, s)$.

Fig. 5 shows the structure of negative β-encoder. For each $x \in [0,1)$ an assignment of bits, i.e. values $b_i^{N\beta} := b_i^{N\beta}(x) \in 0, 1$. Given x, we define $u_1 = \beta x$, and we set $b_1^{N\beta} = 1$ if $u_1 > \nu$, $b_1^{N\beta} = 0$ if $u_1 \le \nu$, i.e., $b_1^{N\beta} = Q_\nu(u_1)$. We then proceed recursively for $i \ge 1$, defining $u_{i+1} = s(b_i^{N\beta}\beta + \overline{b_i^{N\beta}}) - u_i$ and $b_{i+1}^{N\beta} = Q_\nu(u_{i+1})$ where $\overline{b_i^{N\beta}} = 1 - b_i^{N\beta}$. Let $b_i^{N\beta}$ for $i = 1, 2, \cdots, L \in \mathbb{N}$ be a binary expansion of $R(x)$ for $x \in [0,1)$, then

$$R^L(x) = s\sum_{i=1}^{L}(b_i^{N\beta}\beta + \overline{b_i^{N\beta}})(-\beta)^{L-i} + (-\beta)^L x \tag{26}$$

$$\text{or } x = (-\gamma)^L R^L(x) - s\sum_{i=1}^{L}(b_i^{N\beta}\beta + \overline{b_i^{N\beta}})(-\gamma)^i. \tag{27}$$

Since $R^L(x) \in [0, s)$, the midpoint of the interval should be

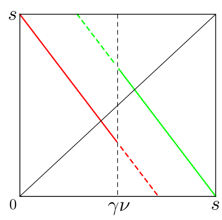

Fig. 4. negative β-map.

chosen as the decoded value of x defined by

$$\tilde{x}_L^{N\beta} = s[(-\gamma)^L/2 - \sum_{i=1}^{L}(b_i^{N\beta}\beta + \overline{b_i^{N\beta}})(-\gamma)^i]. \quad (28)$$

This implies the approximation error between sampled value x and its decoded value $\tilde{x}_L^{N\beta}$ is bounded by $s\gamma^L/2$, which is identical to the one of β-encoder when $s = (\beta-1)^{-1}$ as shown in Fig. 6. However, the variance of negative β-encoder is improved when ν is set to around greedy value and lazy value even if it fluctuates. Similarly, borrowing Daubechies' idea and using the negative β-expansion sequences $b_i^{N\beta}$ for x and $c_i^{N\beta}$ for $y = 1 - x$ $(i = 1, 2, \cdots, L)$, we can get the characteristic equation of β in negative β-encoder as follows:

$$\begin{aligned} P_{N\beta}(\gamma) &= s\{d_1^{N\beta} + \sum_{i=1}^{L-1}(d_{i+1}^{N\beta} - e_i^{N\beta})(-\gamma)^i \\ &+ (1 - e_L^{N\beta})(-\gamma)^L\} - 1 = 0, \end{aligned} \quad (29)$$

where $d_i^{N\beta} = b_i^{N\beta} + c_i^{N\beta}$ and $e_i^{N\beta} = \overline{b_i^{N\beta}} + \overline{c_i^{N\beta}}$.

Fig. 7 shows the error between β and the estimated value $\tilde{\beta}$, root of the characteristic equation of β.

V. Conclusion

The scale adjusted map was introduced to design the amplification factor of β-encoder for given bit budget, tolerance of quantizer, and scale. Futhermore, we introduced the negative β-encoder using the negative real number as the cardinal number which improved the variance of the quantization errors of β-encoder between sampled values and decoded values.

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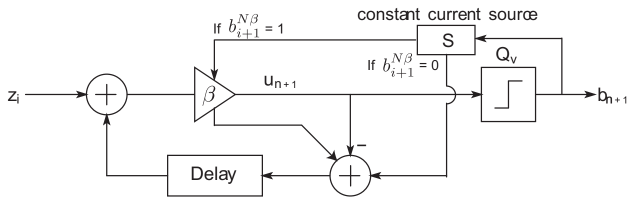

Fig. 5. negative β-encoder: With input $z = x \in [0, 1)$; $z = 0$ for $i > 0$ and "initial conditions" $u_0 = b_0^{N\beta} = 0$ the output $b_i^{N\beta}$ of this block diagram gives the negative β-representation for x defined by the quantizer Q_ν with $\nu \in [s(\beta-1), s]$. $\nu = s(\beta-1)$ is the "greedy" scheme, $\nu = s$ is the "lazy" scheme.

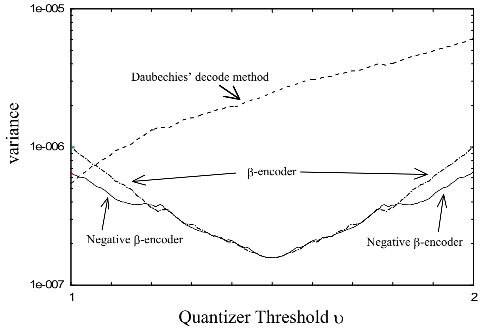

Fig. 6. The variance of the quantization errors when $\beta = 1.5$ and the scale $s = 2.0$ (L=16).

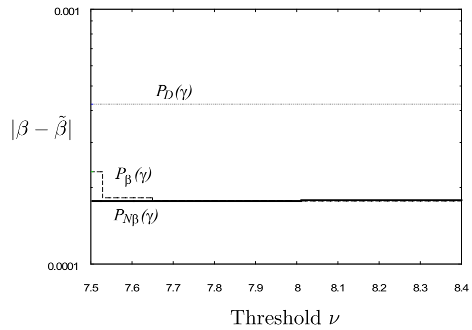

Fig. 7. The error between the amplification factor β and the estimated value $\tilde{\beta}$ when $\beta = 1.8823\cdots$ and the tolerance $\sigma_{\beta,s} = 1.0$ (L = 16).

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