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# A graph Laplacian based approach for facilitating synchronization in networked dynamical systems

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**Abstract**—Collective spontaneous behavior of networked dynamical systems and their structures of networks closely relate to each other. In this paper, we focus on the networked dynamical systems in which nodes are classified into two types, leaders and followers. In this system, the states of the leaders are not affected by other nodes, and they facilitate the synchronization of the followers. We here propose a method for reducing time required for reaching to a synchronous state by selecting leaders based on the eigenvectors of the graph Laplacian and show its performance by numerical simulations.

## 1. Introduction

Coupled dynamical systems are one of effective tools to understand collective spontaneous behavior in several real systems. In particular, the synchronization attracts great attention from several fields, because the synchronous behavior is widely observed in several real technological and biological systems [1–6], for example collective opinion or consensus formation in social systems, synchronization of reproduction among plants in biological systems, and synchronous behavior in a power grid to maintain a steady power supply. From the technological and biological points of view, to facilitate the synchronization is one of important issues.

In this paper, focusing on the facilitation of the synchronization, we propose a method for reducing the time that the networked dynamical systems reach to a synchronous state in which all nodes in the network follow the same trajectory. We start with a simple linear consensus model described by using the graph Laplacian which is a matrix that reflects the coupling topology. We next propose the method for facilitating the synchronization based on the graph Laplacian and show that our method effectively reduces the time required for reaching to the synchronous state of the consensus dynamics. We further show that our method also works well in the case of coupled Rössler systems used as an example of coupled nonlinear dynamical systems.

## 2. Graph Laplacian and networked dynamical systems

The graph Laplacian appears in the several contexts of the network science. Let $A = (a_{ij})$ be the $n \times n$ adjacency matrix of a given network, where $a_{ij} = 1$ if the node $i$ connects to $j$, $a_{ij} = 0$ otherwise. The graph Laplacian of the network is defined by

$$L = K - A. \tag{1}$$

The diagonal matrix $K$ consists of degrees of nodes, namely $K = \mathrm{diag}(k_1, \ldots, k_n)$, where the degree of the node $i$ is described by $k_i = \sum_{j=1}^{n} a_{ij}$. In this paper, we simply assume that all networks are undirected graphs ($a_{ij} = a_{ji}$), and each node does not connect to itself ($a_{ii} = 0$).

Consensus in a networked individual units indicates that a certain quantity that depends on the state of all units reaches to a state of agreement [2]. The consensus dynamics describes how each unit interacts with its neighbors on the network. A simple consensus model in the discrete-time is described as the following linear dynamical system consisting of $n$ nodes

$$\boldsymbol{u}(t+1) = (I - \epsilon L)\boldsymbol{u}(t), \tag{2}$$

where $\boldsymbol{u}(t) = (u_1(t), \ldots, u_n(t))^{\mathsf{T}}$, $u_i(t) \in \mathbb{R}$ is the state of the node $i$ at time $t$, $\mathbb{R}$ is a set of real numbers, $I$ is the $n \times n$ unit matrix , and $\epsilon \in [0, 1/\max_i(k_i))$ is the temporal step size.

On the other hand, in the case of coupled nonlinear dynamical systems, a system consisting of $n$ identical dynamical systems interacting with each other through a connected network is described by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{F}(\boldsymbol{x}_i) - \gamma \sum_{j=1}^{n} L_{ij} \boldsymbol{C}(\boldsymbol{x}_j), \qquad (3)$$

where $\boldsymbol{x}_i$ is the multidimensional state vector of the $i$th dynamical system, $\gamma$ is a coupling strength, $\boldsymbol{F}$ determines the intrinsic dynamics of isolated individual dynamical systems, and $\boldsymbol{C}$ is the coupling function that defines the dynamics of links. Equation (3) covers wide variety of coupling forms and thus the graph Laplacian performs a crucial role in estimating the stability of the coupled dynamical systems [5].

In both models described by Eqs. (2) and (3), if the sum of squares of the differences between all pairs of the state vectors of connected nodes crosses a certain threshold $\theta$, we consider that consensus is reached, or a synchronous state is achieved in this paper. The sum of squares $\boldsymbol{\phi}(t)$ is described by using the graph Laplacian

$$\boldsymbol{\phi}(t) = X^{\mathsf{T}} L X, \qquad (4)$$

where the matrix $X$ consists of the $n$ state vectors of nodes $X = (\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_n(t))^{\mathsf{T}}$. In the case of the consensus model described by Eq. (2), $X = \boldsymbol{u}(t) = (u_1(t), \ldots, u_n(t))^{\mathsf{T}}$.

In this paper, we focus on the networked dynamical systems with leaders-followers [2, 4]. In this model, there exist $m$ $(< n)$ leaders that are the nodes whose states are the same as the states of other leaders. The states of the leaders are not affected by the other non-leader nodes (followers), but nodes connected to the leaders are affected by the leaders. For example, in the consensus model with leaders-followers, Eq. (2) is rewritten by

$$\boldsymbol{u}(t+1) = \left[ I + \epsilon \begin{pmatrix} L_{ff} & L_{lf} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right] \begin{pmatrix} \boldsymbol{u}_f(t) \\ \boldsymbol{u}_l(t) \end{pmatrix}, \qquad (5)$$

where $\boldsymbol{u}(t) = (\boldsymbol{u}_f(t)^{\mathsf{T}}, \boldsymbol{u}_l(t)^{\mathsf{T}})^{\mathsf{T}}$, $\boldsymbol{u}_f(t) = (u_{\rho_1}(t), \ldots, u_{\rho_{n-m}}(t))^{\mathsf{T}}$ is the state vector of the followers, $\boldsymbol{u}_l(t) = (u_{\rho_{n-m+1}}(t), \ldots, u_{\rho_n}(t))^{\mathsf{T}}$ is the state vector of the leaders, the $(n-m) \times (n-m)$ matrix $L_{ff}$ represents the interaction between the follows, and the $(n-m) \times m$ matrix $L_{lf}$ represents the connections from the leaders to the follows. The index of the node $i$ is relabeled by $\rho_i$ $(i = 1, \ldots, n)$ such that the graph Laplacian is partitioned into two block matrices $L_{ff}$ and $L_{lf}$. In the same manner as the consensus model, classifying nodes into two types, namely the leader and the follower, and relabeling the index of the node $i$ as a new index $\rho_i$, we can partition the graph Laplacian in Eq. (3) as follows

$$L = \begin{pmatrix} L_{ff} & L_{lf} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \qquad (6)$$

In the networked dynamical systems with leaders-followers, it is expected that the presence of leaders reduces time when the consensus is reached or the synchronization is achieved.

## 3. Method

In our method, selecting leaders by using the eigenvector corresponding to the second smallest eigenvalue of the graph Laplacian, we propose a method for facilitating the synchronization. From the perspective of the community detection method, the number of links $R$ between two groups, or communities, is described by the graph Laplacian as follows [7]

$$\begin{aligned} R &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(1 - s_i s_j)}{2} a_{ij} \\ &= \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j (k_i \delta_{ij} - a_{ij}) \\ &= \frac{1}{4} \boldsymbol{s}^{\mathsf{T}} L \boldsymbol{s}, \end{aligned} \qquad (7)$$

where $\delta_{ij}$ is the Kronecker delta, namely if $i = j$, $\delta_{ij} = 1$, otherwise zero, $\boldsymbol{s} = (s_1, \ldots, s_n)^{\mathsf{T}}$ is an index vector in which $s_i = 1$ if the node $i$ belongs to one group, but $s_i = -1$ if the node $i$ belongs to another group. The relation $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} s_i^2 k_i = \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j k_i \delta_{ij}$ is used in the second equality. If the nodes $i$ and $j$ belong to the same group, then the term $(1 - s_i s_j)/2$ is zero, but unity if not, and thereby the number of links between two groups are calculated by Eq. (7). By using eigenvalues $\lambda_i$ $(\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_n)$ and their eigenvectors $\boldsymbol{v}_i$ of $L$, Eq. (7) is rewritten by

$$R = \frac{1}{4} \boldsymbol{s}^{\mathsf{T}} L \boldsymbol{s} = \frac{1}{4} \sum_{i=1}^{n} \lambda_i (\boldsymbol{v}_i^{\mathsf{T}} \boldsymbol{s})^2. \qquad (8)$$

By minimizing $R$, nodes in a network can be classified into two communities. If $\boldsymbol{s}$ is proportional to $\boldsymbol{v}_2$, Eq. (8) is minimized, because the eigenvector $\boldsymbol{v}_1$ corresponding to $\lambda_1$ $(= 0)$ is $\mathbf{1}$ and this vector does not give any division of the network. However, under the situation that $s_i$ only takes $-1$ or $1$, it is not possible to define $\boldsymbol{s}$ as the vector proportional to $\boldsymbol{v}_2$. One of possible ways to determine $s_i$ is to set $s_i = 1$ if $v_{i2} \geq 0$ and $s_i = -1$ if $v_{i2} < 0$ such that Eq. (8) is minimized as much as possible. In addition, allowing $s_i$ to take an arbitrary real number, we obtain the relaxed objective function of Eq. (8). The relaxed objective function of Eq. (8) is minimized when $\boldsymbol{s} = \boldsymbol{v}_2$. In this case, the sign of $s_i$ represents the group to which the node $i$ belongs, and the absolute value $|s_i|$ is considered as the importance or centrality of the node $i$ in the group. In our strategy, using the importance of each node in the group, namely the absolute values of the elements in the eigenvector corresponding to the second smallest eigenvalue, we select $m$ leaders in a descending order of their values.

## 4. Numerical experiments

To evaluate the performance of our method, we conduct numerical experiments. We first apply our method to coupled linear dynamical systems in which the dynamics of each node obeys the consensus model described by Eq. (2). The topology of networks are generated from the model proposed by Watts and Strogatz (WS model) in which the links in the initial ring-lattice with the degree $k$ is rewired with a probability $p$ [9]. Then we can generate the ring-lattice ($p = 0$), the small-world networks, and the random networks ($p = 1$) from the WS model. In the numerical simulations, the number of nodes in the network generated from the WS model is set to 500, the degree of the initial ring-lattice is ten, and the number of leaders $m$ is five.

We also employ the model proposed by Barabási and Albert (BA model) that can generate networks whose degree distribution obeys a power law [8]. In the BA model, a single new node with $l$ links is repeatedly added to the current network whose initial state is a complete graph consisting of $l$ nodes. In the numerical simulations, the number of nodes in the network generated from the BA model is set to 5,000, $l$ is set to three, and $m$ is ten.

In addition, we use the real networks: a social network of 62 dolphins [11], human relations between 34 members in the Zachary karate club [12], a co-appearance network between 77 characters in the famous novel *Les Miserables* [13], and the neural network of C. elegans [9]. In these real networks, we set $m$ to five.

We next apply our method to the coupled Rössler systems as an example of the coupled nonlinear dynamical systems [6,10]. The $i$th Rössler system obeys the following dynamics

$$\begin{aligned} \dot{x}_i &= -(y_i + z_i) - \gamma \sum_{j=1}^{n} L_{ij} x_j, \\ \dot{y}_i &= x_i + a y_i, \\ \dot{z}_i &= b + z_i (x_i - c), \end{aligned} \tag{9}$$

where $\gamma$ is a coupling strength, $a = b = 0.2$, $c = 7$, $\gamma \in (\alpha_1/\lambda_2, \alpha_2/\lambda_n)$, $\alpha_1 = 0.1232$, and $\alpha_2 = 4.663$ according to Refs. [5, 6]. In the numerical experiments, selecting $m$ leaders from $n$ nodes, we attempt to reduce the time $T$ when $\phi(t)$ crosses a given threshold $\theta$. The states of leaders are set to the average of the states of followers in this paper. The state of the leaders in the consensus model is $\sum_{i=1}^{n-m} u_{\rho_i}(t)/(n-m)$, and that in the coupled Rössler system is $\sum_{i=1}^{n-m} \boldsymbol{u}_{\rho_i}(t)/(n-m)$, where $\boldsymbol{u}_{\rho_i}(t) = (x_{\rho_i}(t), y_{\rho_i}(t), z_{\rho_i}(t))^{\mathsf{T}}$. The threshold $\theta$ is set to $10^{-9}$ through this paper. In the following, we call the time when the $\phi(t)$ crosses the threshold $\theta$ *the time of consensus*. Let $T_L$ be the time of consensus under the situation that the leaders are selected by the proposed method, $T_r$ be the time of consensus under the situation that the leaders are randomly selected, and $T_o$ be the time of consensus under the case without leaders. Investigating the time of consensus $T_L$, $T_r$, and $T_o$, we evaluate how our method affects to the time when the synchronization is achieved.

## 5. Results

Figure 1 shows a typical example of trajectories of $\phi(t)$ obtained from the consensus model whose network topology is a random network generated by the WS model with $p = 1$. Table 1 shows the summary of the time of consensus. In Table 1, $T_L$, $T_r$, and $T_o$ are averaged over 100 trials. From Table 1, the ratio of the time of consensus $T_L$ to $T_o$ is smaller than or equal to unity, and the ratio $T_L/T_o$ is also smaller than the results of the random selection $T_r/T_o$ in most cases. From these results, our method can effectively reduce the time of consensus. In particular, our method works well in the real networks. The networks of dolphins, members in Zachary's karate club, and characters of Les Miserables have community structures according to Refs. [7, 11]. Our method uses the eigenvector $\boldsymbol{v}_2$ corresponding to the second smallest eigenvalue of the graph Laplacian. Because the values of elements in $\boldsymbol{v}_2$ represents an importance of the nodes in the group, the leaders are likely to be arranged onto each group in our strategy, and thereby our method shows higher performance in the real networks having the community structures than the performance in other networks.

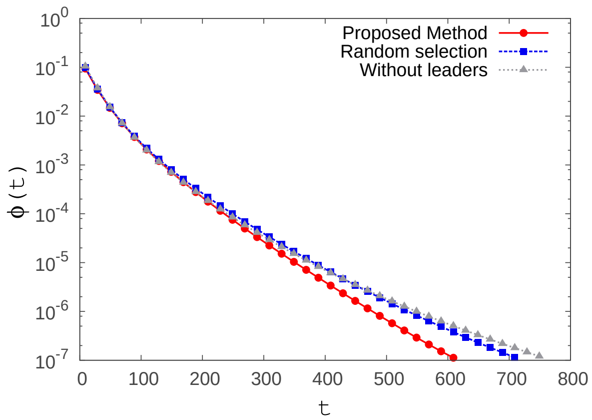


Figure 1: Typical examples of trajectories of $\phi(t)$. The results for the proposed method (red line), the random selection strategy (blue line), and the simple consensus model without leaders-followers (gray line). The number of leaders $m$ is set to five.

Table 1: The results of the time of consensus.

| Dynamics | Network | $T_L/T_o$ | $T_r/T_o$ |
|---|---|---|---|
| Consensus | WS model ($p = 0.1$) | **0.98** | 1.00 |
| Consensus | WS model ($p = 0.5$) | **0.96** | 1.00 |
| Consensus | WS model ($p = 1$) | **0.85** | 0.99 |
| Consensus | BA model | 1.00 | 1.00 |
| Consensus | Dolphins | **0.63** | 0.70 |
| Consensus | C. Elegans | **0.74** | 0.99 |
| Consensus | Zachary's karate club | **0.63** | 0.72 |
| Consensus | Les Miserable | **0.80** | 0.88 |
| Rössler | WS model ($p = 0.1$) | 0.57 | **0.52** |
| Rössler | WS model ($p = 0.5$) | **0.98** | 1.02 |
| Rössler | WS model ($p = 1$) | **0.85** | 0.96 |

## 6. Conclusion

In this paper, we focused on the networked dynamical systems in which nodes are classified into leaders and followers. In the networked dynamical systems with leaders-followers, the states of the leaders are not affected by other nodes, and they facilitate the synchronization of the followers. We proposed a method for reducing time required for reaching to a synchronous state by selecting leaders based on the eigenvectors corresponding to the second smallest eigenvalue of the graph Laplacian. In the numerical experiments, the consensus model and the Rössler systems were arranged onto the nodes in several networks generated from mathematical models and also onto some real networks. As the results, our method works well in most networks that we used in this paper. It is one of important future works to investigate analytically the performance and its limitation of our method analytically.

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