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Common noise-induced synchronization of chaotic oscillators and contraction regions

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Abstract—In this paper, we investigated common noise-induced synchronization of nonidentical chaotic oscillators. We used the Rössler oscillator and the Lorenz oscillator and found that the synchronizabilities in the Rössler oscillator and the Lorenz oscillator are different. To examine why these chaotic oscillators exhibit different synchronizabilities by common noise-induced synchronization, we analyzed contraction regions of these oscillators. As a result, we found that common noise-induced synchronization of nonidentical chaotic oscillators relates to their contraction regions.

1. Introduction

Common noise-induced synchronization has been observed theoretically and experimentally in various nonlinear dynamical systems: for example, neural systems[1, 2], laser systems[3] and so on. It has also been shown that common noise-induced synchronization can be observed in a general class of limit-cycle oscillators[4]. Common noise-induced synchronization using the chaotic oscillators has also been numerically investigated. For example, Zhou and Kurths numerically investigated the noise-induced synchronization of chaotic oscillators[5]. They used the Rössler equation[6] and the Lorenz equation[7] and showed that identical Lorenz systems[7] can perfectly synchronize when large common white Gaussian noise is applied. They also showed that a nonidentical Rössler system exhibits phase synchronization with white Gaussian noise. In our previous study, we experimentally and numerically investigated common noise-induced synchronization[8, 9]. We investigated common noise-induced synchronization of chaotic oscillators using electric circuits[8]. We also performed numerical experiments of common noise-induced synchronization those of mathematical models[9].

In this paper, to investigate why nonidentical chaotic oscillators synchronize by common noise, we focused on the chaotic dynamics. In our numerical experiments, we used the Rössler equation[6] and the Lorenz equation[7] as chaotic oscillators. We clarified that the tendency of synchronization is different in the Rössler oscillator and Lorenz oscillator. In addition, we show why these nonidentical chaotic oscillators are synchronized by common noise.

2. Numerical experiments

In this section, we explain how numerical experiments of common noise-induced synchronization in chaotic oscillators are conducted in this paper. First of all, we defined the noise strength based on signal-noise ratio (SNR) described by the following equation:

$$\mathrm{SNR} = 10\log_{10}\frac{S}{N}, \tag{1}$$

where S is the variance of outputs from chaotic oscillators without applied noise and N is the variance of noise.

In our numerical experiments, we used two uncoupled Rössler oscillators[6] and Lorenz oscillators[7], respectively. The Rössler oscillator is described by the following differential equations:

$$\begin{cases} \dfrac{dx_i}{dt} = -(y_i + z_i) + D_x\xi, \\ \dfrac{dy_i}{dt} = \mu_i x_i + a y_i + D_y\xi, \\ \dfrac{dz_i}{dt} = b + z_i(x_i - c)x + D_z\xi, \end{cases} \tag{2}$$

where a, b, c and μ_i $(i = 1, 2)$ are parameters. The parameters a, b and c in Eq. (2) were set to $a = 0.2$, $b = 0.2$ and $c = 5.0$ in our numerical experiments. The parameter μ_i is a parameter which make the oscillators (Eq. (2)) nonidentical and these parameters are set to $\mu_1 = 1.0$ and $\mu_2 = 0.95$. ξ is a white Gaussian noise whose average and variance are described as: $\langle\xi\rangle = 0$ and $\langle(\xi - \langle\xi\rangle)^2\rangle = 1$. D_N $(N \in \{x, y, z\})$ is a parameter which controls the variance of white Gaussian noise. In our numerical experiments, we applied common noise to the terms of x, y and z, respectively. When we applied the common noise to x, we set the noise strength D_N as $D_y = 0$ and $D_z = 0$. In case of applying common noise to y, we set the noise strength D_N as $D_x = 0$ and $D_z = 0$. In case of applying common noise to z, we set the noise strength D_N as $D_x = 0$ and $D_y = 0$.

We also used the Lorenz equation[7] as a chaotic oscillator. We performed the numerical experiments of the noise-induced synchronization of the Lorenz oscillator described by the following differential equations:

$$\begin{cases} \dfrac{dx_i}{dt} = \rho(-\omega_i x_i + y_i) + D_x\xi, \\ \dfrac{dy_i}{dt} = -x_i z_i + r x_i - y_i + D_y\xi, \\ \dfrac{dz_i}{dt} = x_i y_i - b z_i + D_z\xi, \end{cases} \tag{3}$$

where ρ, r, b and ω_i $(i = 1, 2)$ are parameters. We set these parameters to $\rho = 10$, $r = 28$ and $b = 8/3$. The nonidentical parameter ω_i is set to $\omega_1 = 1.0$ and $\omega_2 = 0.95$. We also used the parameters D_N and ξ which are the same as in the case of the Rössler oscillator. In our numerical experiments of common noise-induced synchronization, we performed Eqs. (2) and (3) using the Euler-Maruyama method[10].

To evaluate the synchronizability of common noise-induced synchronization of these chaotic oscillators, we used cross correlation coefficient. The cross correlation coefficients R_N $(N \in \{x, y, z\})$ is described by the following equations:

$$R_x = \frac{\sigma_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}},\ R_y = \frac{\sigma_{y_1y_2}}{\sigma_{y_1}\sigma_{y_2}},\ R_z = \frac{\sigma_{z_1z_2}}{\sigma_{z_1}\sigma_{z_2}}, \tag{4}$$

where $\sigma_{x_1x_2}$, $\sigma_{y_1y_2}$ and $\sigma_{z_1z_2}$ are covariances of output time series x_i, y_i and z_i $(i = 1, 2)$ of two Rössler oscillators or two Lorenz oscillators, and σ_{x_i}, σ_{y_i} and σ_{z_i} are standard deviations of outputs x_i, y_i and z_i of these chaotic oscillators.

We also focused on the phase difference whether the chaotic oscillators synchronize by common noise. To investigate the phase of the chaotic oscillators and the phase difference between the chaotic oscillators, we used the method based on the Poincaré section. We defined the phase θ_i $(i = 1, 2)$ of two Rössler oscillators and the phase difference $\phi(t)$ as follows:

$$\theta_i(t) = \tan^{-1}\frac{y_i(t)}{x_i(t)}, \tag{5}$$

$$\phi(t) = \theta_1(t) - \theta_2(t), \tag{6}$$

where $x_i(t)$ and $y_i(t)$ are outputs from the Rössler oscillators, respectively.

When we calculated the phase and the phase difference of the Lorenz oscillator, we used the method based on the Poincaré section in Ref. [11]. We defined the phase $\theta_i(t)$ and the phase difference $\phi(t)$ by the following equations:

$$u_i(t) = \sqrt{x_i^2(t) + y_i^2(t)}, \tag{7}$$

$$\theta_i(t) = \tan^{-1}\frac{z_i(t)}{u_i(t)}, \tag{8}$$

$$\phi(t) = \theta_1(t) - \theta_2(t), \tag{9}$$

where $x_i(t)$, $y_i(t)$ and $z_i(t)$ are outputs from the Lorenz oscillator and $u_i(t)$ is a variable to convert the three dimensional space to a two dimensional plane for the Lorenz oscillator. We evaluated the synchronizability using Eqs. (4)-(9).

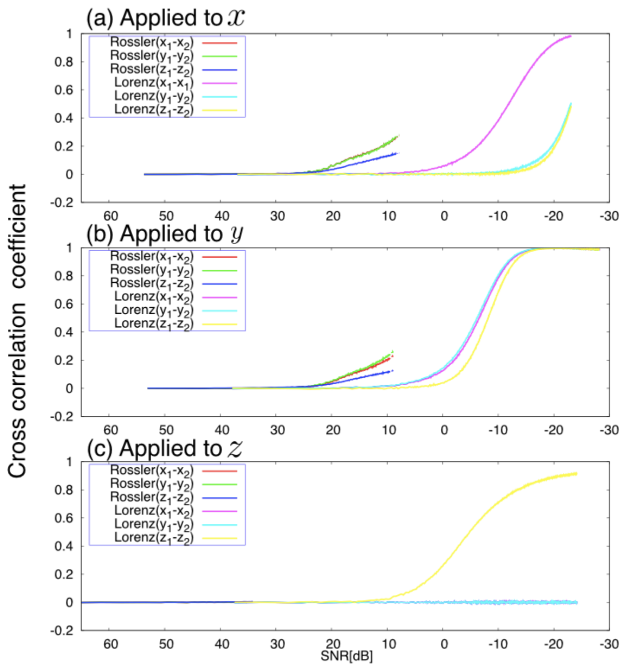

Figure 1: Cross correlation coefficients between the chaotic oscillators. Red, green and blue lines show the results of the Rössler oscillator. Purple, light blue and yellow lines show the results of the Lorenz oscillator. The results of applying common noise to (a) the variable x, (b) the variable y and (c) the variable z.

3. Results

Figure 1 shows the results of the cross correlation coefficients between two chaotic oscillators. Focusing on the results of the Rössler oscillator, we observed that the cross correlation coefficients increase in the case of applying common noise to x and y. However, the maximum value of cross correlation coefficients of the Rössler oscillator is 0.3. It means that only weak synchronization can be observed in the Rössler oscillator. We confirmed that the Rössler oscillator is diverged if the noise level is below 8dB in case of applying common noise to x and y. In case of applying common noise to z, we also confirmed that the Rössler oscillator is diverged with noise level below 34dB.

From the results of the Lorenz oscillator, we found higher cross correlation coefficients than those of Rössler oscillator in all cases (Fig. 1). We also found that the highest cross correlation coefficients are observed in case of applying common noise to y. In case of applying common noise to z, we can also observe high cross correlation only in the output of z.

Figure 2 shows the transition of the peak of the frequency distributions of the phase difference between the chaotic oscillators. We found that the Rössler oscillator shows weak phase synchronization only in the case of applying common noise to x. We also found that the Lorenz

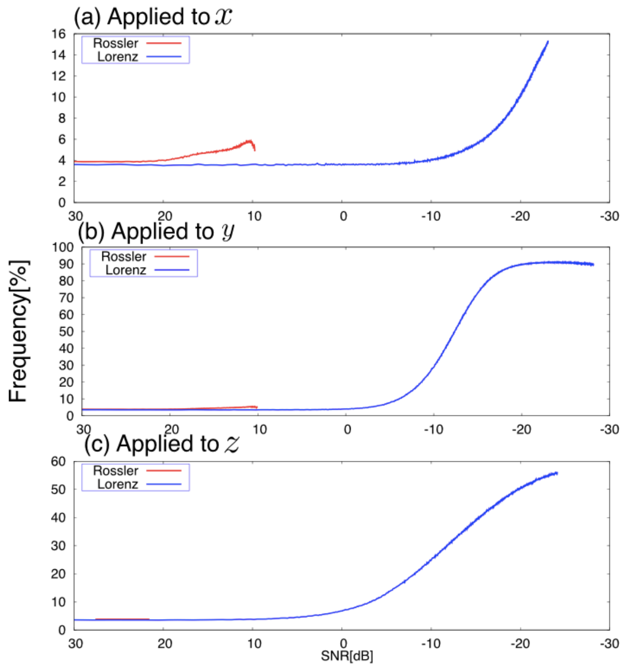

Figure 2: Transition of the peak of the frequency distributions of the phase difference between the chaotic oscillators. Red line shows the results of the Rössler oscillator and blue line shows the results of the Lorenz oscillator. The results of applying common noise to (a) the variable x, (b) the variable y and (c) the variable z.

oscillator shows phase synchronization below 0dB.

Considering these results of cross correlation coefficients and phase synchronization, we can summarize the tendency of noise-induced synchronization of the chaotic oscillators as follows: the Rössler oscillators can be synchronized by smaller common noise than the Lorenz oscillators. However, the maximum synchronizabilty of the Rössler oscillators is low. On the other hand, the Lorenz oscillators can be synchronized very well by common noise. However, to make the Lorenz oscillators synchronize, we need larger noise than that of the Rössler oscillators. We also found from the results of the cross correlation coefficients and the phase synchronization that the highest synchronizability is realized in the case of applying common noise to the output y of the Lorenz oscillator.

We discuss why these chaotic oscillators show the different tendency of synchronization. In Ref. [5], Zhou and Kurths examined contraction region of chaotic oscillators and showed that the contraction regions influenced common noise-induced synchronization. They showed that complete synchronization can be achieved by the contraction region when identical chaotic oscillators are used. Therefore, we investigated how the contraction region affected common noise-induced synchronization of nonidentical chaotic oscillators.

The contraction region is defined as follows: In the contraction region, the real parts of eigenvalues λ_j ($j = 1, \ldots, M$, $M \in \mathbb{N}$) which are calculated by the Jacobian matrix of chaotic oscillators are all negative. In our cases, we calculated the eigenvalues λ_j ($j = 1, 2, 3$) of the Jacobian matrix.

The Jacobian matrix of the Rössler oscillator is described as follows:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}, \qquad (10)$$

where a and c are parameters of the Rössler oscillator and x and z are variables of the Rössler oscillator. The Jacobian matrix of the Lorenz oscillator is described as follows:

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}, \qquad (11)$$

where σ, r and b are parameters and x, y and z are variables of the Lorenz oscillator.

We calculated the contraction region of the chaotic attractors using Eqs. (10) and (11). We also calculated how frequent trajectories of these chaotic attractors exist in their contraction regions.

Table 1 shows the frequency that of the Rössler oscillator and the Lorenz oscillator stay in the contraction regions. In case of the Rössler oscillator, the frequences in the contraction region are low when we do not apply common noise to the Rössler oscillator. In addition, when we applied common noise to various variances of the Rössler oscillator, we cannot observe the increase of the frequency in the contraction region.

In contrast, in case of the Lorenz oscillator, the rate of trajectories within the contraction region is higher than that of the Rössler oscillator even in the absence of noise. By applying noise, we can observe the increase of the rate within the contraction region. In particular, the highest frequency in the contraction region is achieved when applying common noise to y. We also found that the highest synchronizability is realized in case of applying common noise to the output y of the Lorenz oscillator. From these results, it is indicated that common noise-induced synchronization of nonidentical chaotic oscillators relates to those of contraction region.

4. Conclusion

In this paper, we investigated the common noise-induced synchronization of nonidentical chaotic oscillators. We used two uncoupled Rössler oscillator and Lorenz oscillator, respectively. To evaluate the synchronizability of these chaotic oscillators, we used the cross correlation coefficient and the phase difference between these chaotic oscillators. As a result, we found that the tendency of synchronization

Table 1: The frequencies that trajectories of chaotic oscillators exist in their contraction region.

Rössler oscillator			Lorenz oscillator		
Noise applied to	Variance of Noise (SNR)	Average [%]	Noise applied to	Variance of Noise(SNR)	Average
None	-	3.326	None	-	45.771
x	1.5 (10.2dB)	2.453	x	100.0 (-23.1dB)	49.470
y	1.5 (9.6dB)	2.395	y	100.0 (-22.2dB)	79.866
z	0.005 (53.6dB)	3.331	z	100.0 (-22.6dB)	55.377

induced by common noise is different in the Rössler oscillator and the Lorenz oscillator. To investigate this tendency, we investigated the contraction region of these chaotic oscillators. We calculated how frequent the trajectories of these chaotic oscillators stay in their contraction regions. As a result, we found that the common noise-induced synchronization of nonidentical chaotic oscillators relates to their contraction regions.

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