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Noise-Induced Increases in the Duration of Transient Oscillations in Ring Neural Networks and Correlations in the Periods of Ring Oscillators

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Abstract– Effects of external noise on the duration and correlations of oscillations in ring networks of unidirectionally coupled neurons are studied. Changes in the duration of the unstable transient oscillations due to the noise are derived with a kinematical model of traveling waves in the networks. The duration of oscillations occurring from fixed initial conditions increases in the intermediate strength of the noise. Further, positive correlations appear in variations in the periods of stable oscillations in the networks of ring oscillator type.

1. Introduction

Recently, it was shown that long lasting transient oscillations occur in ring networks of unidirectionally coupled neurons in computer simulation [1 - 3] and circuit experiments [4]. The duration of the transient oscillations increases exponentially as the number of neurons in the networks. Such an exponential dependence of transient time on system size has been observed in several systems, e.g. [5]. Such systems never reach asymptotically stable states in a practical time and transient states may play an important role in their function, e.g. signal processing in nervous systems. It is thus of wide interest in the field of nonlinear dynamical systems.

In this study, effects of external noise on the oscillations in the ring neural networks are examined. In Sect. 2, the model networks and the mechanism of the transient oscillations are explained. Changes in the duration of the transient oscillations due to the noise are derived with a kinematical model of the traveling waves in the networks in Sect. 3. Correlations and power spectra of variations in the periods of stable ring oscillators are also shown in Sect. 4.

2. Ring Neural Networks with External Noise

The following ring networks of neurons are considered.

$$\mathrm{d}x_n(t)/\mathrm{d}t = -x_n(t) + f(x_{n-1}(t)) + \sigma_x n_n(t) \quad (x_0 = x_N,\ 1 \le n \le N)$$
$$f(x) = \tanh(gx) \quad (g > 1)$$
$$\mathrm{E}\{n_n(t)\} = 0,\ \mathrm{E}\{n_n(t)n_{n'}(t')\} = \delta_{n,n'}\delta(t - t') \quad (1)$$

where x_n is the state of the nth neuron, N is the number of neurons, $f(x)$ is the output function of the neurons, g is the coupling gain. The neurons are unidirectionally coupled and the output of the Nth neuron is fed backed into the first neuron. Gaussian white noise $n_n(t)$ with the intensity σ_x is added to each neuron independently.

The network has a pair of the non-zero steady points: $(x_1, x_2, \cdots, x_N) = \pm(x_p, x_p, \cdots, x_p)$, $x_p = f(x_p)$ in the absence of noise. It has been shown that it takes a long time for the system to reach one of the steady states when the number of neurons is large, during which the states of neurons oscillate [1 - 4]. In the transient states, the neurons are divided in two blocks in which the signs of their states are the same and the boundaries propagate in the direction of the couplings, e.g. $(x_1, x_2, \cdots, x_N)$: (+, +, +, ⋯, +, -, -, -, ⋯, -) → (-, +, +, ⋯, +, +, -, -, ⋯, -) → (-, -, +, ⋯, +, +, +, -, ⋯, -). The transient oscillations are such traveling waves of the boundaries of the blocks. The velocities of the boundaries depend exponentially on the block length as will be shown in Sect. 3. It then takes an exponentially long time until the two blocks merge so that the network reaches one of the steady states.

3. Changes in Duration of Transient Oscillations

A kinematical model of the traveling boundaries of two blocks in the network in the absence of noise is derived as follows [6]. Consider the state x_{n-1} of the n-1st neuron, and let t_{2j} be the time at which the state x_{n-1} changes from positive to negative and t_{2j+1} be the time at which the state x_{n-1} changes from negative to positive ($j \ge 0$). That is, the two boundaries in the network pass the n-1st neuron at t_{2j}, t_{2j+1} alternately. We here consider the boundary changing from positive to negative sign. The other one can be dealt with in a similar way. When the coupling gain g is large ($g \gg 1$), the output function $f(x)$ is approximated by the sign function (step function). Then the state of the nth neuron changes as follows.

$$\mathrm{d}x_n/\mathrm{d}t = -x_n - 1 \quad (t_{2j} \le t < t_{2j+1})$$
$$x_n(t) = \exp(-(t - t_{2j}))(x_n(t_{2j}) + 1) - 1 \quad (t_{2j} \le t < t_{2j+1}) \quad (2)$$

The propagation time Δt of the boundary from the n-1st to the nth neuron is obtained with $x_n(t_{2j} + \Delta t) = 0$.

$$\begin{aligned}\Delta t &= \log(1 + x_n(t_{2j}))\\ &= \log[1 + \exp(-(t_{2j} - t_{2j-1}))(x_n(t_{2j-1}) - 1) + 1]\\ &= \log[2 - 2\exp(-(t_{2j} - t_{2j-1}))\\ &\quad + \exp(-(t_{2j} - t_{2j-2}))(x_n(t_{2j-2}) + 1)] \quad (3)\end{aligned}$$

We can neglect the second exponential term since its value is exponentially smaller than the first one. The propagation time decreases as the interval to the forward boundary (the length of the forward block) decreases.

The noise term $\sigma_x n_n(t)$ in Eq. (1) gives random variations in the propagation time. They are evaluated with the first passage time (FPT) t_p of the Ornstein-Uhlenbeck (OU) process.

$$dx/dt = -a(x) + \sigma_x n(t) \quad (a(x) = -x)$$
$$E\{n(t)\} = 0, \; E\{n(t)n(t')\} = \delta(t - t')$$
$$x(0) = x_n(t_{2j}) + 1 \approx 2, \quad x(t_p) = 1 \tag{4}$$

where it is assumed that the numbers of neurons in the blocks are large and the neuron fully reaches the steady state after the previous passage of the boundary ($x_n(t_{2j}) \approx x_p \approx 1$). The mean $m(t_p)$ and variance $\sigma^2(t_p)$ of the FPT are given by [7]

$$m(t_p; x(t_p) \mid x(0)) = 2[\int_{x(t_p)}^{x(0)} \pi(\eta)d\eta \int_{\eta}^{B} (\sigma_x^2 \pi(\xi))^{-1} d\xi]$$
$$\sigma^2(t_p) = 4[\int_{x(t_p)}^{x(0)} \pi(\eta)d\eta \int_{\eta}^{B} m(t_p; x(t_p) \mid \xi) / (\sigma_x^2 \pi(\xi))^{-1} d\xi]$$
$$- m(t_p; x(t_p) \mid x(0))^2$$
$$\pi(y) = \exp(-\int^{y} \frac{2a(\eta)}{\sigma_x^2} d\eta) = \exp(\frac{y^2}{\sigma_x^2}), \quad B = \infty \tag{5}$$

Although these integrals are calculated numerically, the mean of the FPT t_p is approximated by the value log2 in the absence of the noise, when the variance of the noise is small ($\sigma_x^2 \ll 1$). The variance can be estimated with the probability density function of $x(t)$.

$$f(x(t) \mid x(0)) = 1/((2\pi)^{1/2}\sigma_p) \cdot \exp[-(x(t) - m_t)^2/(2\sigma_p^2)]$$
$$m_t = x(0)\exp(-t), \quad \sigma_p^2 = \sigma_x^2/2 \cdot (1 - \exp(-2t)) \tag{6}$$

The probability density function of the FPT is approximated by the Gaussian function with the same variance as Eq. (6) since the value of the slope of the trajectory at $x(t) = 1$ ($t = \log 2$) is minus one ($d(2\exp(-t))/dt|_{t = \log 2} = -1$). Hence we obtain

$$m(t_p) \approx \log 2, \quad \sigma^2(t_p) \approx 3/8 \cdot \sigma_x^2 \tag{7}$$

In fact, it can be shown that Eq. (7) agrees with the numerical integral of Eq. (5) for $\sigma_x < 0.1$. The propagation time of the boundary per neuron then becomes

$$\Delta t \approx \log[2 - 2\exp(-(t_{2j} - t_{2j-1}))] + \Delta t_p$$
$$\Delta t_p = t_p - m(t_p) \tag{8}$$

Let b_0 and b_1 be the locations of the two boundaries, and let L be the length of the ring neuron network, i.e. the number of neurons ($L = N$). Further let l and $L - l$ be the length (the number of neurons) of the blocks, i.e.

$$l = b_0 - b_1 \quad (b_0 > b_1)$$
$$= b_0 - b_1 + L \quad (b_0 \le b_1) \tag{9}$$

The propagation velocity of the boundaries b_0 and b_1 is expressed by

$$db_0/dt \approx 1/\Delta t_{2j} = 1/[\log(2 - 2\exp(-(t_{2j} - t_{2j-1}))) + \Delta t_{p,0}]$$
$$db_1/dt \approx 1/\Delta t_{2j+1} = 1/[\log(2 - 2\exp(-(t_{2j+1} - t_{2j}))) + \Delta t_{p,1}] \tag{10}$$

Finally we approximate the intervals $t_{2j} - t_{2j-1}$ and $t_{2j+1} - t_{2j}$ by $(L - l)\Delta t$ and $l\Delta t$ with $\Delta t = \log 2$, respectively, since the difference in them only appear in double exponential terms. By subtracting the second equation from the first equation in Eq. (10), we then obtain the differential equation for the block length l.

$$dl/dt = db_0/dt - db_1/dt$$
$$= 1/[\log 2 + \log(1 - \exp(-\log 2 \cdot (L - l))) + \Delta t_{p,0}]$$
$$- 1/[\log 2 + \log(1 - \exp(-\log 2 \cdot l)) + \Delta t_{p,1}]$$
$$\approx 1/(\log 2)^2 \cdot (\exp(-\log 2 \cdot (L - l)) - \exp(-\log 2 \cdot l)) + \sigma_l n(t)$$
$$\sigma_l^2 = 2\sigma^2(t_p)/(\log 2)^4 = 3/(4(\log 2)^4) \cdot \sigma_x^2 \tag{11}$$

The velocity of the boundary increases as the forward block length decreases and the two boundaries end up by colliding. The oscillation then ceases.

The duration T of the transient oscillations in the presence of noise is dealt with the FPT problem of the stochastic differential equation Eq. (11) with $l(T) = 0$ or L. The mean $m(T(l_0))$ and variance $\sigma^2(T(l_0))$ of the FPT beginning from the initial block length l_0 are then obtained with Eq. (5) by letting

$$a(l) = 1/(\log 2)^2 \cdot [\exp(-\log 2 \cdot (L - l)) - \exp(-\log 2 \cdot l)], \; \sigma_x = \sigma_l$$
$$x(0) = l(0) = l_0, \quad x(t_p) = l(T) = 0, \quad B = L/2 \tag{12}$$

where we use a reflecting boundary at $l = L/2$ owing to the symmetry of the system since the expression is simpler than that with both absorbing boundaries at $l = 0$ and L. Finally we need the following modification owing to the difference between the propagation time with $f(x) = \tanh(gx)$ ($g = 10.0$) and with the sign function. In fact, the duration in the absence of the noise in the computer simulation is as follows.

$$T(l_0 = 15) \approx 28200 \quad (f(x) = \tanh(gx),\, g = 10.0)$$
$$\approx 22800 \quad (f(x)\text{: sign function}) \tag{13}$$

The values of the integral must be multiplied by this ratio (≈ 1.24).

Figure 1 shows the mean $m(T(l_0))$ of the duration of the transient oscillations in the network with $g = 10.0$, $N = 40$ and $l_0 = 15$. Numerical integral of Eq. (5) with Eq. (12) multiplied by 1.24 is plotted with a solid line. The mean of a thousand transient oscillations obtained with computer simulation of Eq. (1) by the simple Euler method with the time step 0.01 under the following initial condition are plotted with closed circles.

$$x_n = -1 \quad (1 \le n \le l_0), \quad x_n = 1 \quad (l_0 < n \le N) \tag{14}$$

The mean duration increases in the intermediate noise strength. This is never seen in the simple OU process and is of particular interest. The approximation by the FPT of Eq. (11) agrees with the simulation results. The decreases of the FPT at small noise strength ($\sigma_x \approx 0.001$) may be an artifact due to the instability of the numerical integral since they do not appear in the simulation.

When the number of neurons is large ($N = L \gg 1$), the FPT problem of simpler form is given by letting exp(-log2·(L - l)) be zero in Eq. (11).

$$\mathrm{d}l/\mathrm{d}t = -1/(\log 2)^2 \cdot \exp(-\log 2 \cdot l) + \sigma_l n(t) \quad (15)$$

It can be shown that the FPT calculated with this equation ($a(l) = -1/(\log 2)^2 \cdot \exp(-\log 2 \cdot l)$) hardly differs from that with Eq. (11) when $N = 40$ and agrees with the simulation results. An intuitive explanation for the increase in the duration due to the noise is as follows. The FPT of Eq. (15) in the absence of the noise is given by $T(l_0) = \log 2 \cdot (2^{l_0} - 1)$. The ratio of the increase $T(l_0 + \Delta l) - T(l_0)$ due to a small positive fluctuation Δl in l_0 and the decrease $T(l_0) - T(l_0 - \Delta l)$ due to a negative fluctuation $-\Delta l$ is $2^{\Delta l}$ and is always larger than one. Then the fluctuations due to the noise tend to increase the FPT. In the simple OU process, the deterministic term is linear and the noise always decreases the FPT. The increase in the duration of the transient oscillations is due to the nonlinear exponential terms in Eqs. (11) and (15).

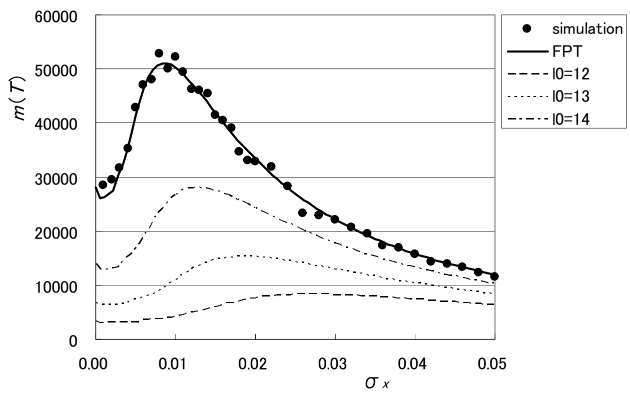

Fig. 1. Mean duration $m(T(l_0))$ of the transient oscillations vs. SD σ_x of the noise.

The peak of the mean duration moves toward large noise strength and the value decreases as the length l_0 of the initial block decreases. Numerical integrals of Eq. (5) with Eq. (12) for l_0 = 12 - 14 are also plotted with dashed, dotted and dash-dotted lines, respectively, in Fig. 1. When starting with $l_0 = N/2$, the oscillations never cease in the absence of the noise since the velocities of the two boundaries are the same. The duration then monotonically decreases as the noise strength increases. When the initial states of neurons are zero or randomly given, the mean duration of the transient oscillations is approximated by the integral of $T(l_0)/N$ over $0 \le l_0 \le N$ since the length of the initial blocks is made uniformly randomly. The duration for the initial block length close to $N/2$ ($l_0 \approx N/2$) mainly contributes to it and it can be shown that the mean duration monotonically decreases as the noise strength.

4. Correlations in Periods of Ring Oscillators

When the number of neurons is odd ($N = 2M + 1$) and the coupling gain is negative and less than the Hopf bifurcation point ($g < -1/\cos(\pi/N)$), the network is equivalent to a ring oscillator and shows stable oscillations [8]. There is one inconsistency in the signs of the states of neurons, e.g. (+, +, -) for $N = 3$, and it rotates in the network as (+, +, -) → (-, +, +) → (+, -, +). We consider variations in half periods of the stable oscillations caused by the noise. The half period is the interval of the passing time of the successive inconsistencies at one neuron and the period of the oscillations is its double.

The propagation time of the inconsistency is derived in a similar way to Sect. 3. The half period T_m of the stable oscillations is given by the following equation.

$$x_m = -(x_m + 1)\exp(-T_m) + 1 \quad (x_m > 0)$$

$$\Delta t_m = \log(x_m + 1), \quad T_m = N\Delta t_m \quad (16)$$

where Δt_m is the propagation time of the inconsistency per neuron in the absence of the noise. The propagation time Δt_j of the jth passing of the inconsistency is given by

$$\Delta t_j = \log(2 - (|x(t_{j-1})| + 1)\exp(-T_j)) + \sigma_t n_j(t)$$

$$\approx \log(2 - (x_m + 1)\exp(-T_j)) + \sigma_t n_j(t)$$

$$\approx \log 2 - \exp(\Delta t_m)/2 \cdot \exp(-T_j) + \sigma_t n_j(t)$$

$$\sigma_t^2 \approx \sigma_x^2/2 \cdot (1 - \exp(-2\Delta t_m))$$

$$T_j = t_j - t_{j-1} \quad (17)$$

Then the changes in the half period T_j at the location l in the network is approximated as

$$\mathrm{d}T_j(l)/\mathrm{d}l = \mathrm{d}(t_j - t_{j-1})/\mathrm{d}l \approx \Delta t_j - \Delta t_{j-1}$$

$$= \exp(\Delta t_m)/2 \cdot (-\exp(-T_j) + \exp(-T_{j-1})) + \sigma_t(n_j - n_{j-1})$$

$$\approx \beta(T_j - T_{j-1}) + \sigma_t(n_j - n_{j-1})$$

$$\beta = \exp(-(N-1)\Delta t_m)/2$$

$$T_j(0) = T_{j-1}(L) \quad (L = N) \quad (18)$$

where n_j is the white noise along t_j. Following [9], the z transform Z_T of T_j is given by

$$\mathrm{d}Z_T(l)/\mathrm{d}l = \beta(1 - z^{-1})Z_T(l) + (1 - z^{-1})Z_n(l)$$

$$Z_T(l) = \exp(\beta(1 - z^{-1})l)Z_T(0) + (1 - z^{-1})\int_0^L \exp(\beta(1 - z^{-1})(l - l'))Z_n(l')\mathrm{d}l'$$

$$Z_T(0) = z^{-1}Z_T(L) \quad (19)$$

Hence the power spectrum $S(\omega)$ of T_j is obtained as

$$S(\omega) = \mathrm{E}\{|Z_T(L)|^2{}_{z = \exp(i\omega)}\}$$

$$= 2(1-\cos(\omega))\sigma_T{}^2 \int_0^L \exp(2\beta(L-l)(1-\cos(\omega)))\mathrm{d}l$$
$$/\,|1-e^{i\omega}\exp(\beta L(1-e^{-i\omega}))|^2$$
$$= \sigma_t{}^2/\beta\cdot(\exp(2\beta L(1-\cos(\omega)))-1)$$
$$/[1+\exp(2\beta L(1-\cos(\omega)))$$
$$-2\exp(\beta L(1-\cos(\omega)))\cos(\omega-\beta L\sin(\omega))] \quad (20)$$

The power spectrum increases in the low frequency region owing to the apparent interaction between the successive half periods since $S(0) = \sigma_t{}^2L/(1 - \beta L)^2$ is larger than $S(\pi) \approx \sigma_t{}^2L$ and $S(\omega) = \sigma_t{}^2L$ for $\beta = 0$.

When $|\beta L| < 1$, a sequence of the half periods is approximated by the first-order autoregressive (AR) process as follows.

$$T_j(l) = t_j(L) - t_{j\text{-}1}(0) = \int_0^L (\beta T_j(l) + \sigma_t n_j)\mathrm{d}l$$
$$\approx \beta L/2\cdot(T_j(0) + T_j(L)) + \sigma_T n_j$$
$$= \varphi_1 T_{j\text{-}1}(0) + \sigma_T n_j$$
$$S(\omega) = \sigma_T{}^2/(1 - 2\varphi_1\cos(\omega) + \varphi_1{}^2)$$
$$\varphi_1 = \beta L/(2 - \beta L), \quad \sigma_T{}^2 = \sigma_t{}^2L/(1 - \beta L/2)^2 \quad (21)$$

where the trapezoidal rule is used for the estimation of the integral. The parameter φ_1 has the positive value and the autocorrelation function $\mathrm{E}\{(T_j(l) - T_m)(T_{j\text{-}k} - T_m)\}$ of the sequence is given by $\sigma_T{}^2/(1 - \varphi_1{}^2)\cdot\varphi_1{}^k$.

Figure 2 shows the power spectrum of the sequences of the half periods in the inhibitory network (g = -10.0) of three neurons (N (= L) = 3) in the presence of noise of σ_x = 0.1. An estimate with FFT of the results of computer simulation is plotted with closed circles. Equations (20) and (21) are also plotted with solid and dashed lines, respectively, where the values of the parameters are: $x_m = (-1 + 5^{1/2})/2 \approx 0.618$, $\Delta t_m = \log((1 + 5^{1/2})/2) \approx 0.481$, $T_m = \log(2 + 5^{1/2}) \approx 1.44$, $\beta = (3 - 5^{1/2})/4 \approx 0.19$, $\sigma_t = (-1 + 5^{1/2})^{1/2}/2\cdot\sigma_x \approx 0.056$, $r \approx 0.40$, $\sigma_T \approx 0.13$. The approximation with Eqs. (20) and (21) agrees about with the simulation results though the correlations are slightly large.

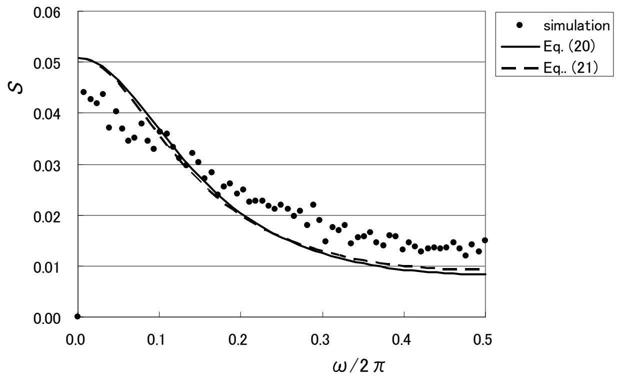

Fig. 2. Power spectrum $S(\omega)$ of a sequence T_j of the half periods in the ring oscillator of three neurons.

These correlations occur in the interspike intervals of a spike propagating in a ring of excitable media [9]. The propagation time of the spike increases as the interspike interval decreases in the refractory period of the excitable media. The value of the coefficient β is then negative and the interspike intervals become negatively correlated. It is of interest that the effects of the interaction in the ring are opposite to those of the interspike intervals of a spike train propagating in a line of excitable media, e.g. a nerve fiber. The interaction then smooths the interspike intervals and cause positive correlations when $\beta < 0$. Similarly to a spike train in the nerve fiber, when an input is added to one end of an open chain of the sigmoidal neurons, signals switching positive and negative signs can be generated and propagated toward the other end. Then the interaction with $\beta > 0$ may increase variations in the intervals of switching and make them negatively correlated, which results in a modulation of the signals.

5. Conclusion

It was shown that the duration of the transient oscillations in the unidirectionally coupled ring neural networks increases in the intermediate noise strength on the basis of the FPT problem of the kinematical model of the traveling waves. This noise-induced resonance like phenomenon is due to the exponential dependence of the velocities of the traveling waves on the forward block length of the neurons.

Further, it was shown that the noise causes positive correlations in the sequences of the periods of the stable oscillations in the network of ring oscillator type. The power spectrum is expressed with not rational but exponential functions of frequency while the sequences can be approximated by the 1st-order AR process.

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