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Properties of the Duration of Transient Oscillations in a Ring Neural Network

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Abstract- Properties of the duration of long lasting transient oscillations in ring networks of unidirectionally coupled sigmoidal neurons are derived with a kinematical model of traveling waves in the networks. The duration of the transient oscillations occurring from random initial conditions increases exponentially as the number of neurons. The distribution of the duration is approximated by a power-law function when the number of neurons is large.

1. Introduction

An exponential dependence of transient time on system size is of wide interest in the field of nonlinear dynamical systems [1 - 5]. In such systems, transient time from initial states to stable states increases exponentially as the size of systems. These systems never reach their asymptotically stable states in a practical time when the system size is sufficiently large. Their functions, e.g. information processing in the nervous systems, may proceed in the transient states, as has been pointed out in the references. The transient states thus play more important roles than the asymptotic states in actual systems.

In this study we consider transient oscillations in ring networks of sigmoidal neurons. Although the model considered here is simple, it has been studied for recurrent neural networks [6] and as cyclic feedback systems [7]. It is then known that a discrete-time version of it has multiple stable orbits [8]. Further, it has been shown that, if delays exist, similar models cause various spatio-temporal patterns [9, 10] and long lasting transient oscillations [11 - 13].

When the number of inhibitory couplings is odd, the ring network is qualitatively the same as a ring oscillator which outputs stable rectangular waves. When the number of inhibitory couplings is even, on the other hand, it is known that the steady states are globally stable and stable oscillations never exist [6, 7, 13 - 15]. Recently, however, it was shown that the networks of even inhibitory couplings have oscillations lasting for a very long time [16 - 18]. Further, oscillations continuing about a thousand times before ceasing were observed in experiments on analog circuits [19].

In the following, we derive a kinematical model of traveling waves in the ring neural networks. It is then shown that the duration of the transient oscillations increases exponentially as the number of neurons and is distributed in a power-law form.

2. Ring Neural Network and Transient Oscillations

The following ring network of unidirectionally coupled neurons is considered.

$$\tau \mathrm{d}x_n/\mathrm{d}t = -x_n + f(x_{n-1}) \quad (x_0 = x_N,\ 1 \le n \le N)$$
$$f(x) = \tanh(gx) \quad (g > 1) \qquad (1)$$

where x_n is the state of the nth neuron, N is the number of neurons, $f(x)$ is the output function of the neurons, g is the coupling gain and τ is the time constant. The neurons are coupled to the adjacent neurons unidirectionally and their output is transmitted through the monotonically increasing nonlinear function f with the gain g. The network has a pair of the non-zero steady points.

$$x_n = x_p = f(x_p) \quad (1 \le n \le N)$$
$$x_p \to \pm 1 \text{ for } |g| \to \infty \qquad (2)$$

Although long lasting transient oscillations were shown originally in the networks of inhibitory couplings ($g < -1$) we use excitatory couplings ($g > 1$) for simplicity. In fact, the networks with an even number of neurons and negative couplings are equivalent to those of positive couplings by changing as $x_{2m} \to -x_{2m}$ ($1 \le m \le N/2$), $g \to -g$. Further, when there are both of excitatory and inhibitory couplings in the networks, they can be transformed into networks of all positive couplings if the number of inhibitory couplings is even by appropriately changing the signs of the states and the couplings.

Figure 1 shows an example of temporal patterns in the states of neurons in the transient oscillations beginning from random initial states $x_n(0)$, in which black and white regions correspond to the states of positive and negative signs, respectively. After a short time the neurons are separated in two propagating blocks in which the states of neurons have the same signs. The transient oscillations are these traveling waves of the blocks. After a long transient time the two blocks merge and the oscillation ceases so that the neurons reach one of the steady states.

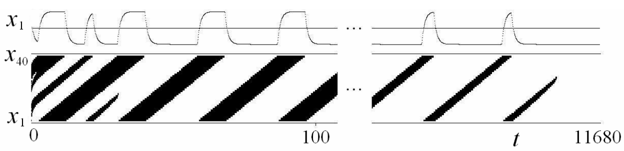

Fig. 1. Temporal patterns of transient oscillation.

3. Kinematical Model of Traveling Waves

We derive a kinematical model of the traveling waves which correspond to the transient oscillations. Let two blocks of the same signs of the states of neurons be made in the network. Consider the state x_{n-1} of the n-1st neuron, and let t_{2j} be the time at which the state x_{n-1} changes from positive to negative and t_{2j+1} be the time at which the state x_{n-1} changes from negative to positive ($j \geq 0$). That is, the two boundaries of the two blocks in the network pass the n-1st neuron at t_{2j}, t_{2j+1} alternately. We call the boundaries of 2jth and 2j+1st passing time the p-n and n-p boundaries, respectively. The input to the nth neuron changes its sign at this time. When the coupling gain is large ($g \gg 1$), the output function $f(x)$ is approximated by the sign function (step functions). Then the state x_n of the nth neuron changes as follows.

$$\tau dx_n/dt = -x_n - 1 \quad (t_{2j} \leq t < t_{2j+1})$$
$$= -x_n + 1 \quad (t_{2j+1} \leq t < t_{2j+2}) \tag{3}$$

The solutions of Eq. (3) are given by

$$x_n(t) = \exp(-(t - t_{2j})/\tau)(x_n(t_{2j}) + 1) - 1 \qquad (t_{2j} \leq t < t_{2j+1})$$
$$= \exp(-(t - t_{2j+1})/\tau)(x_n(t_{2j+1}) - 1) + 1 \quad (t_{2j+1} \leq t < t_{2j+2}) \tag{4}$$

Let Δt_{2j} be the elapsed time by which the p-n boundary propagates from the neuron n - 1 to the neuron n, i.e. $x_n(t_{2j} + \Delta t_{2j}) = 0$. The propagation time Δt per neuron is expressed by

$$\Delta t_{2j} = \tau\cdot\log(1 + x_n(t_{2j}))$$
$$= \tau\cdot\log(1 + [\exp(-(t_{2j} - t_{2j-1})/\tau)(x_n(t_{2j-1}) - 1) + 1])$$
$$= \tau\cdot\log(2 - 2\exp(-(t_{2j} - t_{2j-1})/\tau) + \exp(-(t_{2j} - t_{2j-2})/\tau)(x_n(t_{2j-2}) + 1)) \tag{5}$$

The expression for the propagation time Δt_{2j+1} of the n-p boundary is given in the same way. When the length of the blocks is large, we can approximate the propagation time by $\Delta t = \tau\cdot\log 2$ since the intervals $t_{2j} - t_{2j-1}$ and $t_{2j} - t_{2j-2}$ in the exponential terms are large. The period of the oscillations is then about $N\Delta t = \tau N\log 2$.

To obtain a closed form, we neglect the second exponential term in the right-hand side of Eq. (5) since it is exponentially smaller than the first exponential term owing to that the interval in the exponential is twofold. We then let b_0 and b_1 be the locations of the p-n and n-p boundaries, respectively, and let L be the length of the ring neuron network, i.e. the number of neurons ($L = N$). Further let l be the length (the number of neurons) of one block and $L - l$ be that of the other block, i.e.

$$l = b_0 - b_1 \qquad (b_0 > b_1)$$
$$= b_0 - b_1 + L \quad (b_0 \leq b_1) \tag{6}$$

The propagation velocities of the boundaries b_0 and b_1 are expressed by

$$db_0/dt \approx 1/\Delta t_{2j} = 1/[\tau\cdot\log(2 - 2\exp(-(t_{2j} - t_{2j-1})/\tau))]$$
$$db_1/dt \approx 1/\Delta t_{2j+1} = 1/[\tau\cdot\log(2 - 2\exp(-(t_{2j+1} - t_{2j})/\tau))] \tag{7}$$

Finally we approximate the intervals $t_{2j} - t_{2j-1}$ and $t_{2j+1} - t_{2j}$ by $(L - l)\Delta t$ and $l\Delta t$ with $\Delta t = \tau\cdot\log 2$, respectively, since the difference in them only appear in double exponential terms. By subtracting the second equation from the first equation in Eq. (7), we then obtain the differential equation for the length l of the block.

$$dl/dt = db_0/dt - db_1/dt$$
$$= 1/[\tau(\log 2 + \log(1 - \exp(-\log 2\cdot(L - l))))] - 1/[\tau(\log 2 + \log(1 - \exp(-\log 2\cdot l)))]$$
$$= 1/[\tau(\log 2 + \log(1 - 2^{-(L - l)}))] - 1/[\tau(\log 2 + \log(1 - 2^{-l}))] \tag{8}$$

4. Properties of Duration of Transient Oscillations

4.1 Dependence of the duration on the initial block length

To derive an explicit form of $l(t)$, we approximate Eq. (8) by

$$dl/dt \approx 1/[\tau(\log 2 - 2^{-(L - l)})] - 1/[\tau(\log 2 - 2^{-l})]$$
$$\approx 1/\tau\cdot(1/\log 2 - 2^{-(L - l)}/(\log 2)^2 - (1/\log 2 - 2^{-l}/(\log 2)^2))$$
$$= k(\exp(-c(L - l)) - \exp(-cl))$$
$$k = 1/[\tau(\log 2)^2], \; c = \log 2 \tag{9}$$

We can set $l \leq L/2$ without any restrictions, i.e. we regard l as the length of the smaller block. The duration T of the transient oscillations is then given as $l(T) = 0$, which means the boundaries merge and the smaller block disappears at T since the all x_n then converge to x_p or - x_p quickly.
The solution of Eq. (9) can be obtained by substituting $y = \exp(cl)$.

$$\exp(cl(t)) = \exp(cL/2)\tanh(-\exp(-cL/2)ckt + \operatorname{arctanh}(\exp(cl_0 - cL/2)))$$
$$l_0 = l(0) \quad (0 \leq l_0 \leq L/2) \tag{10}$$

The duration T of the transient oscillations are given by setting $l(T) = 0$.

$$T = 1/(ck)\cdot\exp(cL/2)[\operatorname{arctanh}(\exp(c(l_0 - L/2))) - \operatorname{arctanh}(\exp(-cL/2))] \tag{11}$$

The duration T increases exponentially as the total length L. Further, it increases exponentially as the initial length l_0 of the smaller block, i.e. the number of neurons in the block, when L is large since $\operatorname{arctanh}(x) \approx x$ for small x.

A simpler form of the duration $T(l_0)$ for large L is derived by letting L be infinity in Eq. (9). The equation and the solution become

$$\mathrm{d}l/\mathrm{d}t = -k\exp(-cl)$$

$$l(t) = 1/c\cdot\log(\exp(cl_0) - ckt)$$

$$T = (\exp(cl_0) - 1))/ck = \tau\log2\cdot(2^{l_0} - 1) \quad (l(T) = 0) \qquad (12)$$

where $k = 1/[\tau(\log2)^2]$, $c = \log2$ and $l_0 = l(0)$ $(0 \le l_0 \le L/2)$. This shows the exponential dependence of the duration T on the initial block length l_0. Further, the block length decreases in proportion to time in the beginnings, i.e. $l_0 - l(t) \approx 2^{-l_0}/[\tau(\log2)^2]\cdot t$ for $t \ll \tau(\log2)^2 2^{l_0}$, and the decrease rate decreases exponentially as the initial length.

Figure 2 shows results of computer simulation with Eq. (1) under the following initial conditions

$$x_n = -1 \quad (1 \le n \le l_0)$$
$$= 1 \quad (l_0 < n \le N) \qquad (13)$$

with g = 10.0, the time step: 0.01 and N (= L) = 40. The duration T of the transient oscillations was measured with the conditions $|x_1(T) + f(x_N(T))| < 0.0001$ and $|x_n(T) - f(x_{n-1}(T)| < 0.0001$ for $2 \le n \le N$. The duration increases exponentially as the initial block length. Equations (11) and (12) well agree with the simulation results.

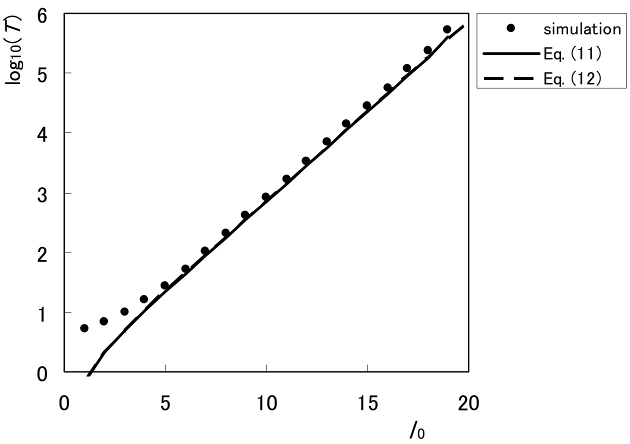

Fig. 2. Duration T of the transient oscillations with the initial block length l_0 $(1 \le l_0 \le 19)$ and $N = 40$.

4.2 Distribution of the duration under random initial conditions

When the initial values $x_n(0)$ $(1 \le n \le N)$ of the states of neurons are given randomly, the initial length of the smaller block is considered to be distributed uniformly $(0 \le l_0 \le L/2)$. Then the probability density function $h(T)$ of the duration T of the transient oscillations is derived from

$$\int_0^{l_0} U(0, L/2)\mathrm{d}l_0 = \int_0^T h(T')\mathrm{d}T' \qquad (14)$$

where $U(a, b)$ is the uniform distribution between a and b. Hence we obtain

$$h(T) = |\mathrm{d}T(l_0; L)/\mathrm{d}l_0|^{-1}/(L/2)$$
$$= 2k\exp(-cL/2)\mathrm{cosech}[2(\exp(-cL/2)ckT + \mathrm{arctanh}(\exp(-cL/2)))]\cdot 2/L$$
$$= 2/[\tau(\log2)^2]\cdot 2^{-L/2}\mathrm{cosech}[2(2^{-L/2}T/[\tau\log2] + \mathrm{arctanh}(2^{-L/2}))]\cdot 2/L \qquad (15)$$

There is a cut-off point $T_c = \tau\log2\cdot 2^{L/2}$ at which the form of the probability density function changes. On one hand, for $T > T_c$ or when the total length L (the number N of neurons) is large, the approximate form is derived by using $\mathrm{arctanh}(\varepsilon) \approx \varepsilon$ and $\sinh(\varepsilon) \approx \varepsilon$

$$h(T) \approx k/(ckT+1)\cdot 2/L = 1/[\tau(\log2)^2(T/(\tau\log2) + 1)]\cdot 2/L$$
$$(0 \le T \le 1/ck\cdot(\exp(cL/2) - 1))$$
$$= \tau\log2\cdot(2^{L/2} - 1)) \qquad (16)$$

This is also derived from Eq. (12), which is obtained from Eq. (9) with L infinity. The duration T is thus distributed in the form of $1/T$. It is of interest that such a power-law distribution appears in this simple nonlinear system. On the other hand, for $T < T_c$ or when the total length L is small, the form is approximated by the exponential distribution by using $\sinh(x) \approx \exp(x)/2$ $(x \gg 1)$ for large T in Eq. (15)

$$h(T) \approx \lambda\exp(-\lambda T)$$
$$\lambda \approx 2\exp(-cL/2)ck = 1/[2^{L/2-1}\cdot\tau(\log2)] \qquad (17)$$

The cut-off point increases exponentially as the number of neurons and the region in which the duration is distributed in the power-law form extends.

Figure 3 shows the probability density function h of the duration T of the oscillations in the network with the numbers N (= L) of neurons 10 (a), 20 (b), 40 (c). Plotted are a normalized histogram of the duration of 10000 transient oscillations obtained with computer simulation of Eq. (1) (closed circles) and Eqs. (15) - (17) (solid, dashed, dotted lines, respectively). The initial states $x_n(0)$ is drawn from Gaussian random numbers with the mean 0 and SD 0.1 in the simulation. Note that Fig. 3(a), (b) are semi-log plots and Fig. 3(c) is a log-log plot. The histogram in Fig. 6(c) is composition of those made per decade. When N = 10 (a), the histogram of the simulation results decreases about exponentially as the duration and is approximated by Eq. (17). When N = 20 (b), the histogram deviates from the exponential distribution and a long tail appears. Equation (15) gives the best fit to this intermediate form. When N = 40 (c), the slop of the log-log graph of the histogram is close to -1 and the density decreases as the inverse of the duration. The power law form is retained in a wide range $(10^1 < T < 10^6)$. The cut-off point $T_c \approx 7.3\times10^5$ in Eq. (15) agrees with it.

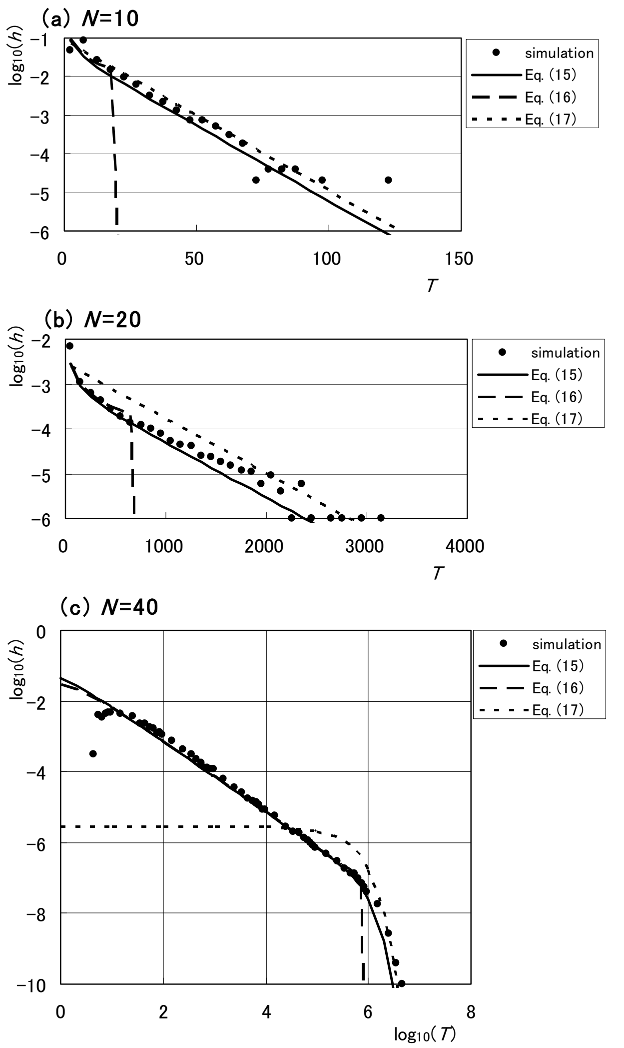

Fig. 3. Probability density function h of the duration T of the transient oscillations in the network with the numbers N (= L) of neurons 10 (a), 20 (b), 40 (c).

The mean m, the variance σ^2 and the coefficient of variation CV of the duration T are calculated with $h(T)$. Using Eqs. (12) and (16) for large L, they are expressed explicitly as

$$m(T(L)) = 2(\exp(cL/2) - 1 - cL/2)/(c^2kL)$$

$$= 2\tau(2^{L/2} - 1 - \log 2/2 \cdot L)/L$$

$$\sigma^2(T(L)) = (\exp(cL) - 4\exp(cL/2) + 3 + cL)/(c^3k^2L) - \{m(T(L))\}^2$$

$$\mathrm{CV}(T(L)) = \sigma(T(L))/m(T(L))$$

$$\approx (cL)^{1/2}/2 = (\log 2)^{1/2}/2 \cdot L^{1/2} \quad (L \gg 1) \tag{18}$$

Thus the mean and SD (σ) increase exponentially as the number (L) of neurons and they are multiplied by $2^{1/2}$ ($\approx 10^{0.15}$) per neuron. The relative variation (CV) increases as the square root of the number of neurons and is larger than 1 for $L \geq 10$.

5. Conclusion

The kinematical model of the traveling waves in the ring network of unidirectionally coupled neurons was derived. The velocity of the traveling wave exponentially increases as the forward block length and the oscillations last exponentially long. The mean duration of the oscillations occurring from random initial conditions increases also exponentially as the total number of neurons in the network ($m(T) \sim 2^{N/2}$). Further, the distribution of the duration of the transient oscillations changes from the exponential distribution to the power-law form as the number of neurons increases.

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