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# Exponential Transient Oscillations and Standing Pulses in Rings of Coupled Symmetric Bistable Maps

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**Abstract-** The author studies transient states the duration of which increases exponentially with system size (exponential transients) in rings of coupled symmetric bistable maps. When coupling is unidirectional (one-way coupling), transient oscillations rotating in a ring of maps exist. When coupling is bidirectional (two-way coupling), transient pulse patterns exist. In both coupling the duration of transient states on the way to spatially homogeneous steady states increases exponentially with the number of elements. Further, the duration of transient states occurring from random initial states is distributed in a power law form.

## 1. Introduction

Exponential transients are transient states the duration (lifetime) of which increases exponentially with system size. They were first found in a one-dimensional bistable reaction-diffusion system, which is known as the time-dependent Ginzburg-Landau equation in the field of phase transitions [1]. In one-dimensional domain, a transient front (kink), a pulse (kink-antikink pair) and multiple pulse patterns are formed until a system reaches a spatially homogeneous steady state. The motion of these patterns extremely slow, which is called metastable dynamics, and the duration of them increases exponentially with domain length. Such dynamically metastable patterns have since been studied in various reaction-diffusion equations and convection-diffusion equations in multi-dimensional domains [2].

Another form of exponential transients is transient chaos or supertransients, which was found in coupled map lattices [3]. A lattice shows complex turbulent patterns retaining invariant statistical measures and suddenly falls onto a periodic attractor. The duration of complex behaviors also increases faster than exponentially with the number of elements. Such transient chaos has since been studied in various systems including reaction-diffusion equations, excitable media and complex networks [4].

Recently, exponential transient oscillations with the same mechanism as metastable dynamics were found in a unidirectionally coupled ring of sigmoidal neurons [5]. Qualitatively the same transient oscillations were also found in a discrete-time system, i.e. a ring of directly coupled bistable cubic maps [6]. These exponential transient patterns with dynamical oscillations are of interest since they lie between simple static spatial transient patterns and complicated spatiotemporal chaotic transient patterns.

In this study, the author considers exponential transients in rings of symmetric bistable maps with standard coupling. In unidirectionally coupled maps, traveling pulses rotating in a ring similar to those in a ring neural network arise in its transient state. In bidirectionally coupled maps, transient pulse patterns like metastable patterns in reaction-diffusion systems exist when coupling is strong and nonlinearity is weak. It is shown that both transient states last exponentially long time with the number of elements and that the duration of them under random initial conditions obeys a power-law distributions up to a cut-off.

## 2. Rings of Coupled Bistable Maps

The following rings of coupled maps are considered.

$$x_n(t+1) = (1-\varepsilon) f(x_n(t)) + \varepsilon f(x_{n-1}(t)) \tag{1a}$$

$$x_n(t+1) = (1-\varepsilon) f(x_n(t)) + \varepsilon/2\{f(x_{n-1}(t)) + f(x_{n+1}(t))\} \tag{1b}$$

$$f(x) = x + K\sin(2\pi x)/(2\pi)$$
$$(1 \le n \le N, \quad x_0 = x_N, \quad 0 < \varepsilon < 1, \quad 0 < K < 2 \tag{2}$$

where coupling is unidirectional in Eq. (1a) with strength ε and is bidirectional in Eq. (1b) with strength ε/2. Figure 1 shows a symmetric bistable map $f(x)$ with some values of a parameter $K$ of nonlinearity. The map as well as the rings have bistable fixed points $x_n = \pm 1/2$ and au unstable one $x_n = 0$.

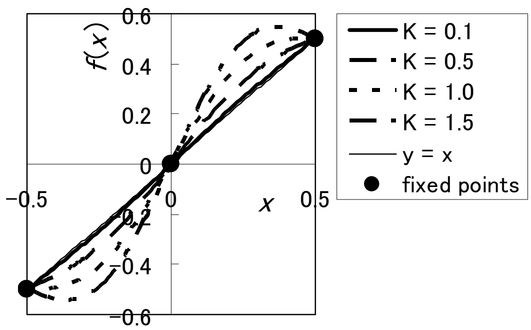



Fig. 1. Symmetric bistable maps.

Figure 2(a)(b) shows examples of transient oscillations occurring under random initial conditions with a uniform distribution: $x_n(0)$~ U(-1/2, 1/2) in Eq. (1a) (unidirectional coupling) with $N = 20$, ε = 0.2 (a), 0.8 (b) and $K = 0.5$. Time courses of the states $x_1(t)$ of the first elements are plotted in upper panels, and spatiotemporal patterns of the states of elements are plotted with black (white) for positive (negative) signs in lower two panels. Single traveling pulses are quickly generated from random initial states, while they are unstable and the states converge to one ($x_n$ = -1/2) of spatially homogeneous bistable states after a long time. The speed of a traveling wave is slow when coupling strength ε is small (a), and vice versa (b).

Figure 2(c) shows an example of transient pulses in Eq. (1b) (bidirectional coupling) with $N = 40$, ε = 0.5 and $K = 0.1$, in which snapshots of the states of elements at $t$ = 0, 100, 5000, 10000, …, 35000 (by 5000) are plotted. A standing pulse is also generated quickly, keeps its form for a long time, and finally collapses.

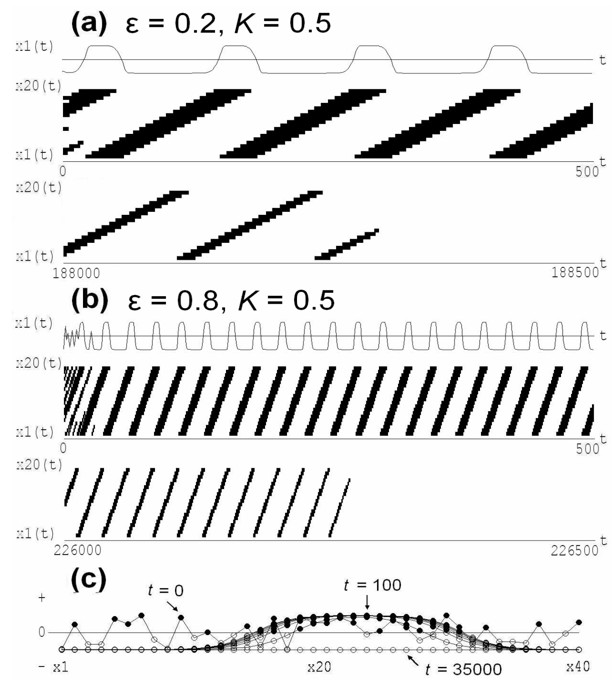



Fig. 2. Spatiotemporal patterns in transient states for unidirectional coupling (a) (b), bidirectional coupling (c).

## 3. Transient Oscillations with Unidirectional Coupling

### 3.1. Kinematics of Traveling Waves

When coupling is unidirectional and the number of elements is even: $N = 2l_h$, an unstable aperiodic (quasi-periodic) traveling wave exists in the subspace: $x_n = x_{n+N/2}$ $(1 \le n \le l_h)$. It is a symmetric pulse wave with equal pulse widths and is observed with computer simulation of Eq. (1a) under a symmetric initial condition:

$$x_n = c^{1/2} \quad (1 \le n \le l_0), \quad x_n = -c^{1/2} \quad (l_0 < n \le N) \qquad (3)$$

where $l_0$ and $N$ - $l_0$ are initial spatial pulse widths, i.e. the numbers of elements in pulses, and both are set to be a half $l_h$ (= $N$/2) of the number of elements. The speed of the symmetric traveling wave depends on its spatial pulse width $l_h$. We define the propagation time (an inverse of the speed) of the traveling wave by time required for the propagation of pulse fronts over one unit distance (one element). Figure 3 shows a semi-log plot of $\Delta t(l_h)$ - $\Delta t_\infty$ obtained with computer simulation of Eq. (1a) under Eq. (3) with ε = 0.2 and $K = 0.5$ against $l_h$, where $\Delta t_\infty = \Delta t(l_h = 8)$ is propagation time for $l_h$ large enough. Difference in the propagation time decreases exponentially with $l_h$, hence the number $N$ of elements. It is approximated by

$$\Delta t(l_h) = \Delta t_\infty + b\exp(-al_h) \qquad (l_h = N/2)$$
$$a \approx 2.375, \quad b \approx 52.5, \quad \Delta t_\infty \approx 6.345 \qquad (4)$$

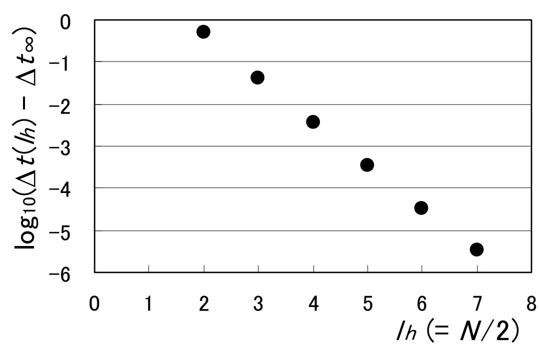



Fig. 3. Propagation time of traveling waves.

Let us assume that the propagation time of each pulse front depends on its *backward* pulse width as was done in [6]. Changes in the spatial width $l$ of one pulse are then expressed by difference between the inverses of the propagation times of two pulse fronts:

$$\mathrm{d}l/\mathrm{d}t \approx 1/\Delta t(l) - 1/\Delta t(N-l)$$
$$= -\beta\{\exp(-\alpha l) - \exp[-\alpha(N-l)]\} \qquad (5)$$
$$\alpha = a \approx 2.375, \quad \beta = b/\Delta t_\infty^2 \approx 1.304$$

It should be noted that the propagation time of a pulse fronts depends on the forward pulse width in a ring neural network [5]. The mechanism causing the dependence of the propagation time of a front on the backward pulse width in coupled map lattices is not clear at present.

### 3.2. Properties of Transient Oscillations

The solution $l(t)$ of Eq. (5) with initial pulse width $l(0) = l_0$ is obtained as

$$\exp(-\alpha l(t)) = \exp(\alpha N/2)\tanh\{-\exp(-\alpha N/2)\alpha\beta t + \mathrm{arctanh}[\exp(\alpha(l_0 - N/2))]\} \qquad (6)$$

The duration $T$ of the traveling wave and transient oscillation is given by letting $l(T) = 0$:

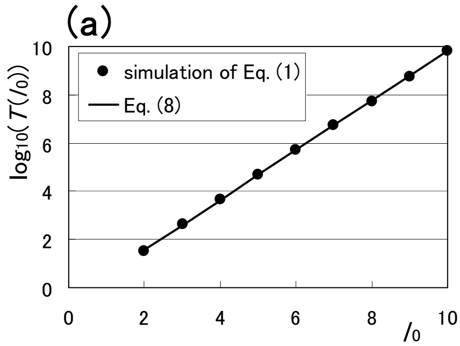
$$T(l_0;N) = \frac{\exp(\alpha N/2)}{\alpha\beta}\{\mathrm{arctanh}[\exp(\alpha(l_0 - N/2))] - \mathrm{arctanh}[\exp(-\alpha N/2)]\} \qquad (7)$$

Simpler forms of Eqs. (6) and (7) are given by letting $N$ be infinity in Eq. (5):

$$\begin{aligned} &\mathrm{d}l/\mathrm{d}t = -\beta\exp(-\alpha l) \\ &l(t) = 1/\alpha\cdot\log[\exp(\alpha l_0) - \alpha\beta t] \quad (l(0) = l_0 < N/2) \\ &T(l_0) = [\exp(\alpha l_0) - 1]/(\alpha\beta) \quad (l(T) = 0) \end{aligned} \tag{8}$$

Figure 4(a) shows a semi-log plot of the duration $T$ of transient oscillations against initial pulse width $l_0$ in Eq. (1a) with $N$ = 21, ε = 0.2 and $K$ = 0.5. Plotted are the results of computer simulation of Eq. (1) under Eq. (3) (solid circles) and $T(l_0)$ in Eq. (8) (a dashed line). The duration increases exponentially with initial pulse width, and Eq. (8) agrees with the simulation results.

Figure 4(b) shows a semi-log plot of the duration $T$ of transient oscillations against coupling strength ε obtained by computer simulation of Eq. (1a) under Eq. (3) with $N$ = 20 and three sets of ($K$, $l_0$). One of interest is symmetry in $T$ with respect ε = 1/2 despite the fact that the propagation time of a pulse front monotonically increases with ε as shown in Fig. 1. The other is that unstable asymmetric traveling waves are stabilized into stationary standing pulses as ε → 0, while they become stable traveling waves as ε → 1. That is, the duration $T$ diverges near both sides of graphs in Fig. 4(b).



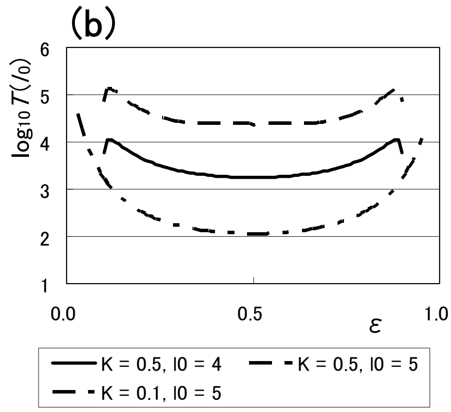




Fig. 4. Duration of oscillations vs initial pulse width $l_0$ (a) and vs coupling strength ε (b).

Further, the distribution $h(T)$ of the duration $T$ of transient oscillations occurring from random initial states is obtained by letting initial pulse width $l_0$ be distributed uniformly in {0, $N$/2} with $T(l_0; N)$ in Eq. (7):

$$\int_0^{l_0} U(0, N/2)\mathrm{d}l_0' = \int_0^T h(T')\mathrm{d}T' \tag{9}$$

$$\begin{aligned} h(T) &= \frac{1}{|\mathrm{d}T(l_0; N)/\mathrm{d}l_0|}\cdot\frac{2}{N} = \left|\frac{\mathrm{d}l_0(T; N)}{\mathrm{d}T}\right|\cdot\frac{2}{N} \\ &= 4\beta\exp(-\alpha N/2)\mathrm{cosech}\{2[\exp(-\alpha N/2)\alpha\beta T \\ &\quad + \mathrm{arctanh}(\exp(-\alpha N/2))]\}/N \end{aligned} \tag{10}$$

There is a cut-off $T_c$ in $h(T)$ in Eq. (10), and the duration is distributed in an inverse power law form for $T < T_c$:

$$h(T) = \frac{\beta}{\alpha\beta T + 1}\cdot\frac{2}{N} \qquad (0 < T < T_c) \tag{11}$$

while it is distributed exponentially for $T > T_c$:

$$\begin{aligned} &h(T) \approx 4\lambda\exp(-\lambda T)/(\alpha N) \\ &\lambda \approx 2\alpha\beta\exp(-\alpha N/2) \qquad (T > T_c) \end{aligned} \tag{12}$$

Figure 5 shows a log-log plot of the distribution $h(T)$ of the duration $T$ of oscillations, in which plotted are a histogram obtained with $10^4$ runs of computer simulation of Eq. (1a) with $N$ = 15, ε = 0.2 and $K$ = 0.5 under $x_n(0) \sim$ U(-1/2, 1/2) (solid circles), Eq. (10) (a solid line), Eq. (11) (a dashed line) and Eq. (12) (a dotted line). Equation (10) agrees with the simulation results, and Eq. (11) agrees with them also up to a cut-off: $T_c \approx 1.76\times10^7$.

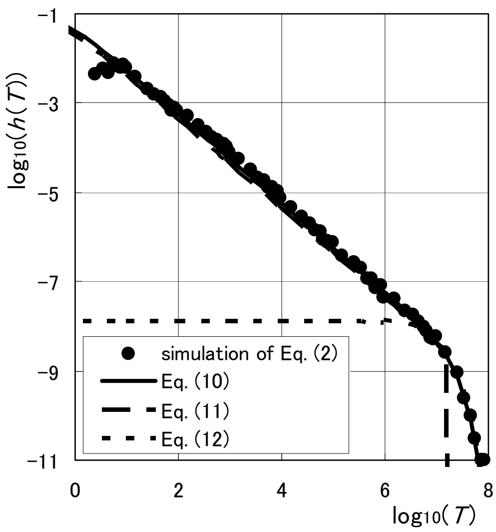



Fig. 5. Distribution of the duration of oscillations.

The mean $m(T)$ and variance $\sigma^2(T)$ of the duration $T$ of oscillations also increase exponentially with the number $N$ of elements:

$$\begin{aligned} &m(T) = 2[\exp(\alpha N/2) - 1 - \alpha N/2]/(\alpha^2\beta N) \\ &\sigma^2(T) = [\exp(\alpha N) - 4\exp(\alpha N/2) + 3 + \alpha N]/(\alpha^3\beta^2 N) \\ &\quad - m(T)^2 \\ &CV(T) = \sigma(T)/m(T) \approx (\alpha N)^{1/2}/2 \end{aligned} \tag{13}$$

and the coefficient of variation ($CV$) increases with the square root of $N$. Figure 6 shows the logarithms of $m(T)$ and σ($T$) as well as $CV(T)$ of the duration against the number $N$ of elements for Eq. (1a) with ε = 0.2 and $K$ = 0.5. Plotted are estimates with $10^4$ runs of computer simulation of Eq. (1a) (symbols) and Eq. (13) (lines), in which they agree with each other.

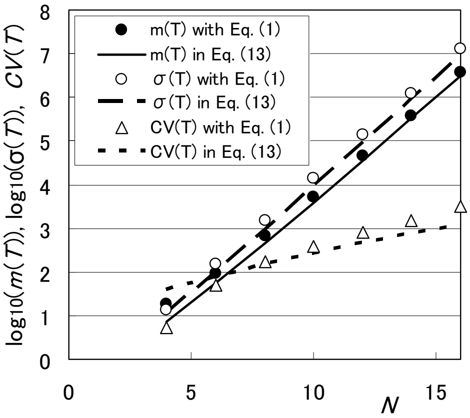



Fig. 6. $m$, σ and $CV$ of the duration of oscillations.

## 4. Transient Pulses with Bidirectional Coupling

When coupling is bidirectional, computer simulation of Eq. (1b) can show that standing pulses are stabilized when pulse width is over a threshold depending on parameter values. It is known that such stabilization of spatially inhomogeneous patterns occurs generally in spatially discrete systems. However, pulses of smaller width than a threshold are still unstable, and the duration of them increases exponentially with pulse width in the same way as that for unidirectional coupling. Figure 7 shows a semi-log plot of the duration $T$ of pulses against initial pulse width $l_0$ obtained by computer simulation of Eq. (1b) under Eq. (3) with $N$ = 100 and three sets of ($K$, ε). The duration increases exponentially, and a threshold length increases as $K$ decreases and ε increases.

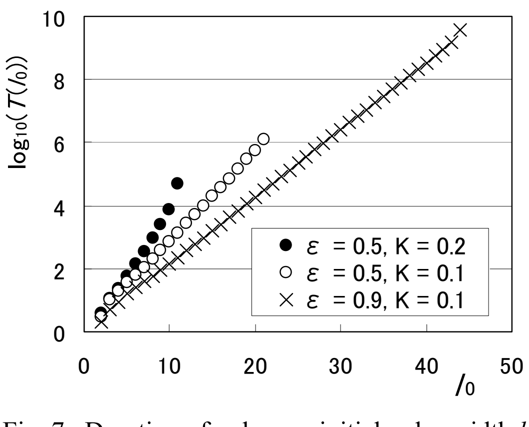



Fig. 7. Duration of pulses vs initial pulse width $l_0$.

Figure 8 shows a log-log plot of the distribution $h(T)$ of the duration of pulses, in which plotted are a histogram obtained with $10^4$ runs of computer simulation of Eq. (1b) under $x_n(0) \sim U(-1/2, 1/2)$ with $N$ = 40, ε = 0.5 and $K$ = 0.1 (solid circles) and Eqs. (10) - (12) (lines). The values of parameters are estimated by fitting $T(l_0)$ in Eq. (8) to $T(l_0)$ in Fig. 7 and are set to be α = 0.651 and β = 0.487. The duration is distributed in an inverse power law form up to a cut-off $T_c \approx 1.14\times10^6$, and Eqs. (10) and (11) agree with the simulation results except for small $T$ (< 10).

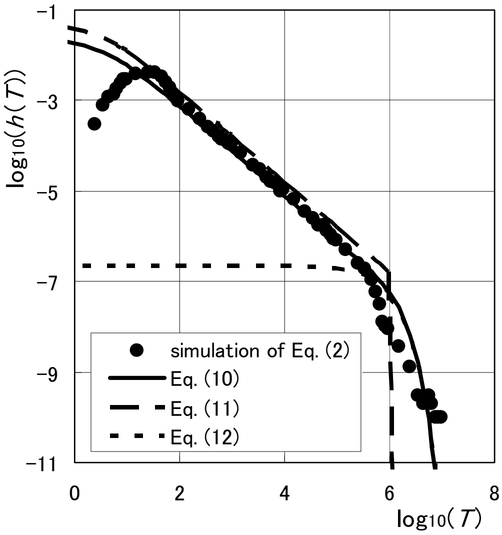



Fig. 8. Distribution of the duration of pulses.

## 5. Conclusion

Properties of transient oscillations in a ring of unidirectionally coupled symmetric bistable maps and transient pulses in a ring of bidirectionally coupled maps were studied. Changes in pulse width were described by qualitatively the same equation (Eq. (5)) as those in reaction-diffusion systems [1] and ring neural networks [5]. The duration of both transient states increased exponentially with pulse width and the number of elements. Further, the duration of them under random initial conditions obeyed a power law distribution function.

Analytical derivation of the kinematics (Eqs. (4) and (5)) of the motion of pulses and analysis of intrinsic duality causing the symmetry in Fig. 4(b) and bifurcations causing changes in the stability of traveling (Fig. 4(b)) and standing (Fig. 7) pulses are future studies.

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