2015 International Symposium on Nonlinear Theory and its Applications
NOLTA2015, Kowloon, Hong Kong, China, December 1-4, 2015

# Wave Packet Propagation on An Array of Switches

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**Abstract**—This paper presents a simple microscopic model of defective conductors. The model consists of particles, particle guides, and switch boxes. The particles and probabilistic switch boxes respectively represent electrons and lattice defects scattering electrons. The model is considered as a probabilistic hybrid system since the scattering is discontinuous change of continuous motion of electrons. We have analyzed packets of probability distribution of the particle location and found that the particles lose their initial momentum at around time $1/(1-p)$, $p$ : probability that the switch boxes do not change particle direction, and diffuse like random walkers.

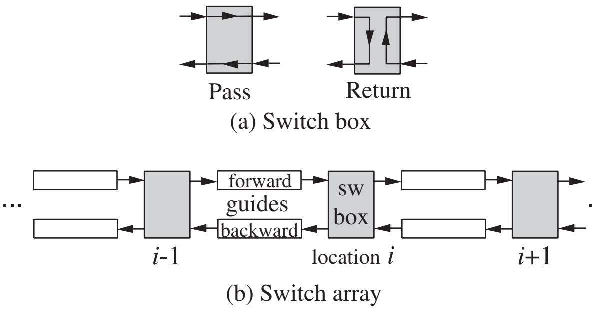


Figure 1: Model structure.

## 1. Introduction

Hybrid systems are defined as systems consisting of both continuous and discontinuous parts. Microscopically, conductors including impurities and lattice defects scattering electrons are considered as hybrid systems since scattering is regarded as discontinuous change of continuous motion of electrons. In this paper, assuming that the collision of electrons against impurities and defects is completely elastic, we will model such conductors with particles representing electrons, particle guides, and switches scattering the particles at certain probability. We then analyze evolving packets of probability distribution of the particle location by referring to [1] and [2]. The model can be applied to the analysis of propagation of sharp pulses on conducting lines.

## 2. A Model and its Description

Figure 1 shows the proposed model. It contains particles all of which move at equal speed. The model is built of equal length guides on which the particles move and probabilistic switch boxes in which the particles pass at probability $p$ or return at probability $1-p$. Pairs of forward and backward guides and the switch boxes are connected alternately. Because of equal particle speed and one-way guides, particles do not collide against each other.

Let $l/v$ ($l$: guide length, $v$: particle speed) be normalized. Then, location $x$ of a particle is described by the following discrete-time equation [1]:

$$x(n+1) = x(n) + \Gamma(n) \tag{1}$$

where $n$ is the integer independent time variable and $\{\Gamma(n)\}$ is a binary random sequence given by

$$\begin{aligned} &\Gamma(n) \in \{+1, -1\}, \\ &\mathrm{Prob}(\Gamma(n) = +1 | \Gamma(n-1) = +1) = \\ &\quad \mathrm{Prob}(\Gamma(n) = -1 | \Gamma(n-1) = -1) = p, \quad 0 < p < 1, \\ &\mathrm{Prob}(\Gamma(n) = +1 | \Gamma(n-1) = -1) = \\ &\quad \mathrm{Prob}(\Gamma(n) = -1 | \Gamma(n-1) = +1) = 1-p, \end{aligned} \tag{2}$$

As $p$ increases/decreases from $1/2$, $\{\Gamma(n)\}$ becomes colored noise and contains more lower/higher frequency components.

Let $P_{\mathrm{F/B}}(i, n)$ denote a probability that a particle is at the left / right input of a switch box at position $i$ at time $n$. Then, evolution of $P_{\mathrm{F/B}}(i, n)$ is described similarly to [2] by

$$P_{\mathrm{F}}(i, n+1) = pP_{\mathrm{F}}(i-1, n) + (1-p)P_{\mathrm{B}}(i+1, n) \tag{3}$$

$$P_{\mathrm{B}}(i, n+1) = (1-p)P_{\mathrm{F}}(i-1, n) + pP_{\mathrm{B}}(i+1, n) \tag{4}$$

Now, the model is regarded as a kind of probabilistic cellular automata [3, 4, 5].

## 3. Numerical Experiments

We compute a probability that a particle is on a guide at position $i$ at time $n$ on a condition that the particle is at the left entrance of a switch box at position 0 at time 0. Figures 2 and 3 show wave packets of probability distributions $P_{\mathrm{F}}(i, n)+P_{\mathrm{B}}(i, n)$ obtained by the numerical integrations of the probabilistic difference equation (1) with (2) for 5000 particles and of the evolutional equation set of the distribution, Eqs. (3), (4), for $p = 0.95$ and 0.99. The obtained distributions are regarded as a dispersion process of an impulsive pulse propagating on a resistive conductor with impurities and defects. Figures 4 (a) and (b) respectively show the expectation of the particle location and the variance of the distribution plotted against the time for $p$ = 0.95, 0.98, and 0.99. The plot points and the lines in the figures are obtained respectively by the numerical integrations of the probabilistic difference equation and of the evolutional equation set.

## 4. Discussion

The continuum approximation of the model transforms difference equation set (3) and (4) to the following differential equation [2]:

$$\left\{(2p-1)\frac{\partial^2}{\partial t^2} + 2(1-p)\frac{\partial}{\partial t} - p\frac{\partial^2}{\partial x^2}\right\} P(x,t) = 0 \qquad (5)$$

where $P$ is $P_\mathrm{F}$ or $P_\mathrm{B}$, $x$ and $t$ are independent space and time variables. When $p = 1$, Eq. (5) expresses lossless wave systems. When $p = 1/2$, Eq. (5) represents diffusion systems with no advective term.

We extend the range of $P(x,t)$ from real to complex domain and let it be given by

$$P(x,t) = \exp(\gamma t + jkx), \quad j^2 = -1 \qquad (6)$$

Substituting Eq. (6) into Eq. (5), we obtain the following characteristic equation:

$$(2p-1)\gamma^2 + 2(1-p)\gamma + pk^2 = 0 \qquad (7)$$

with its characteristic exponent given by

$$\begin{aligned} &\gamma = \alpha \pm j\omega, &&(8)\\ &\alpha = -\frac{1-p}{2p-1}, \quad \omega = \frac{\sqrt{(2p-1)pk^2 - (1-p)^2}}{(2p-1)} \end{aligned}$$

When the dissipation of the wave system (5) is very small, that is, the coefficient of the first-order derivative term is $2(1-p) \approx 0$, speed of the wave propagation is $\omega/k = \sqrt{p/(2p-1)}$. When $2(1-p) > 0$, the time constant of attenuation of the wave is $\tau = (2p-1)/(1-p) \approx 1/(1-p)$. At around the time, propagating components almost disappear and dispersed components remain. Then, the particles on the model lose their momentum at around time $n = 1/(1-p)$ and the particle distribution spreads around location $i = (\omega/k)\tau/2 = 1/2/(1-p)$. This estimation agrees with the results shown in Fig. 4(a). When wave packets distribute in space with width $\sigma = k^{-1} = \sqrt{(2p-1)p}/(1-p) \approx 1/(1-p)$, we find $\omega = 0$ from Eq. (8). Then, they can not propagate. The deviation of the distribution of the particles presented in Fig. 4(b) is roughly $\sigma$ at time $\tau$ when the particles lose their momentum.

## 5. Conclusions

We have analyzed packets of probability distribution of the particle location, that is the statistical behavior of the particles, on a hybrid systems consisting of pairs of forward and backward guides and switch boxes. As a result of the analysis, we have found that the particles lose their momentum at around time $n = 1/(1-p)$ and diffuse like random walkers.

## References

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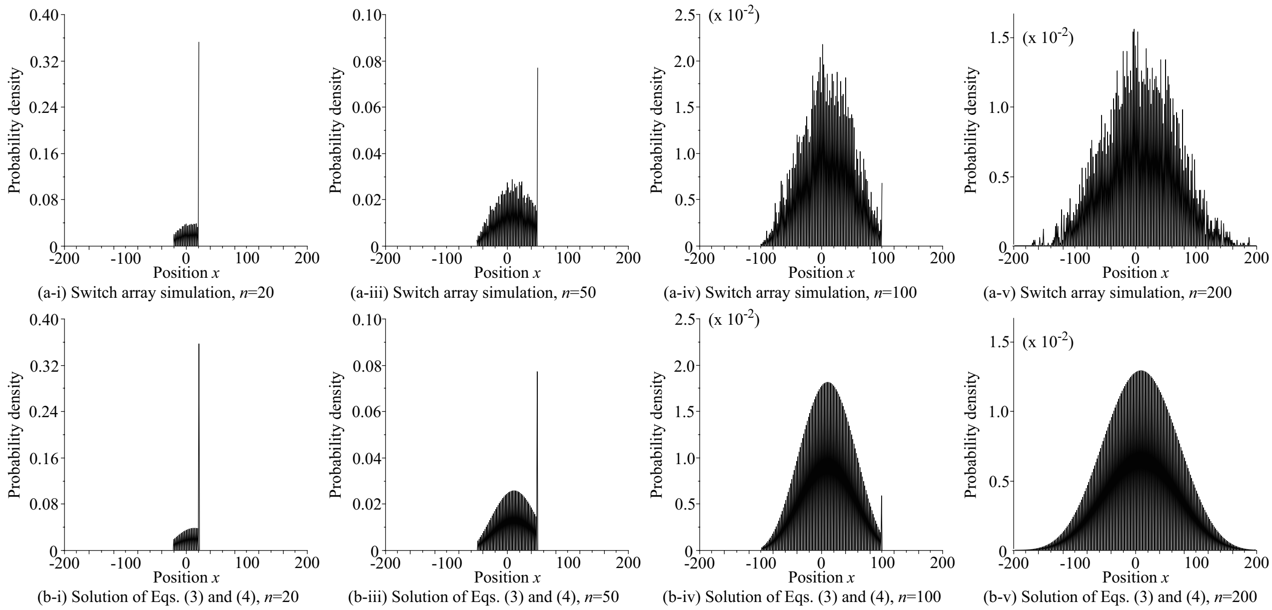


Figure 2: Probability distributions of particle location ($p$ = 0.95).

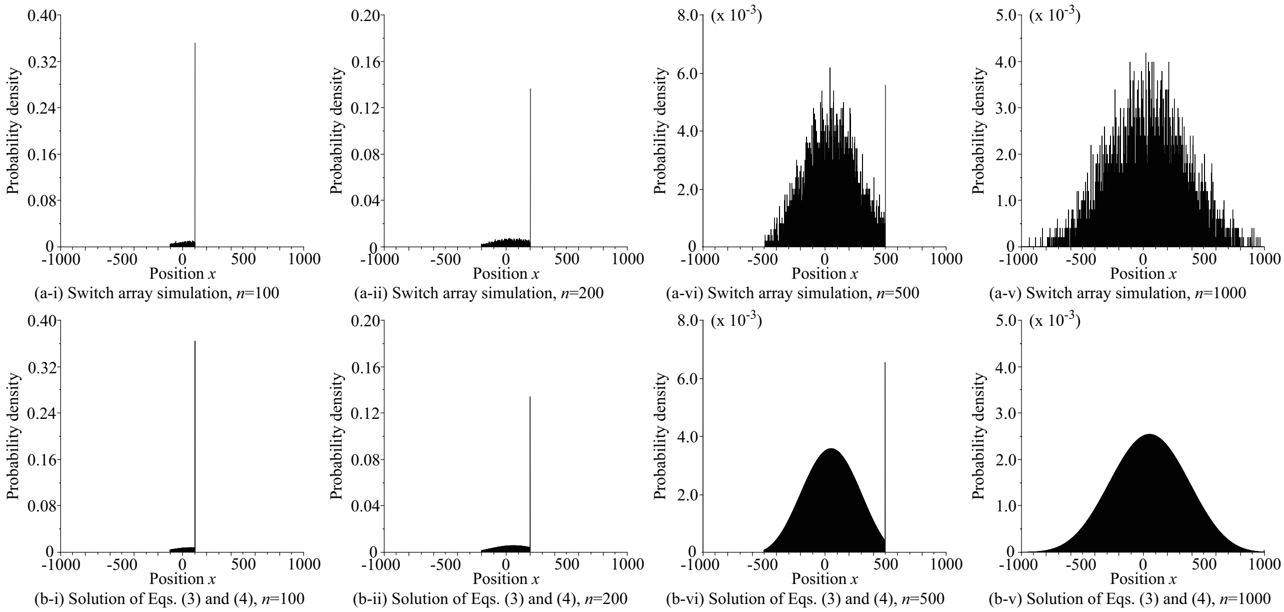


Figure 3: Probability distributions of particle location ($p$ = 0.99).

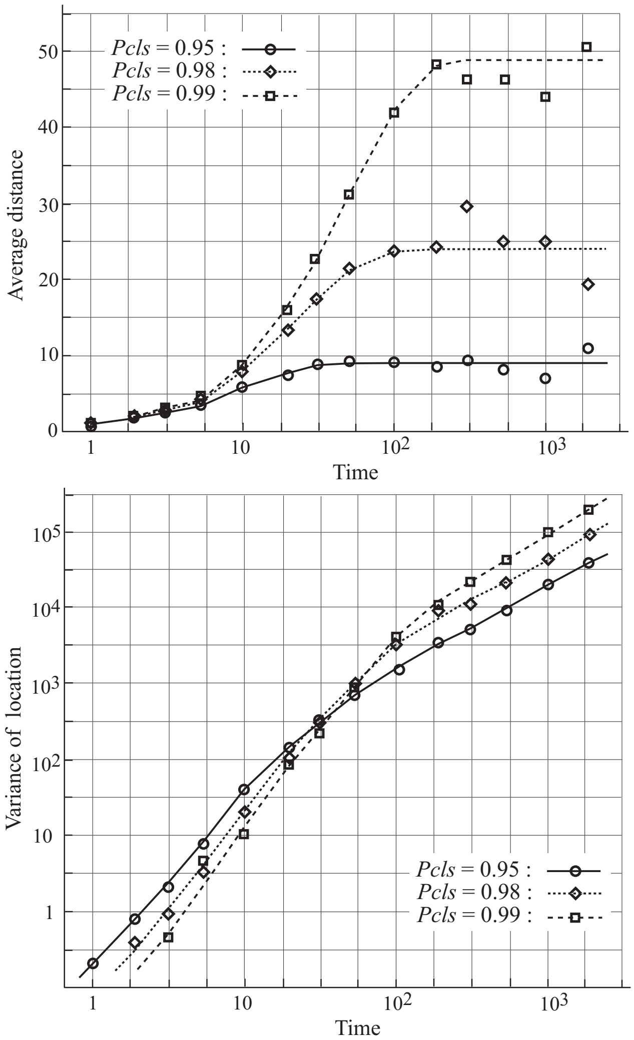


Figure 4: Expectation and variance of the probability distributions of particle location. The plot points and the lines are obtained respectively by the numerical integrations of the probabilistic difference equation (1) with (2) for 5000 particles and the evolutional equation set of the distribution, Eqs. (3), (4).