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Mathematical Modeling of Road Heating System with Underground Distribution Line Based on Nonlinear ODE Model

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Abstract—In regions with heavy snowfall, snow accumulation causes problems such as falls on sidewalks and traffic congestion. In this paper, we propose a road heating system using underground power distribution lines. In this paper, the system is decomposed into four compomnents: voltage distribution in the lines, thermal dissipation in the cable, thermal diffusion in the ground, and snow melting on the ground surface. Based on the equations of each component, we derive a mathematical model of the road heating system considered as a main result. The nonlinear ordinary differential equation (ODE) model is introduced to describe the distribution voltage. The validity of the model is verified by a numerical simulation.

1. Introduction

In heavy snowfall regions, winter snow accumulation causes problems such as falling on sidewalks and traffic congestion. Road heating systems are as an electrical device to solve these problems. The system promotes snow melting by burying a thermal source underground, and raises the temperature of the ground surface. Melting snow with the system is less likely to cause refreezing and is expected to have a stronger effect than sprinkling water.

This paper examines the use of underground power distribution lines as a thermal source for a road heating system. The lines can be regarded as cables used to transmit electric power underground. If the heat generated from the lines can be used to melt snow, we can construct an innovative power and energy system that has multiple functions such as power distribution and snow melting.

The nonlinear ordinary differential equation (ODE) model [1] is a mathematical model that describes the spatial voltage distribution of a power distribution system. In [2], the temperature variation of underground power lines was demonstrated when a periodically varying current is injected to the lines. Although the distribution voltage fluctuates in accordance with the load connected to the line, the temporal change has not been examined in temperature of the line in response to such a situation.

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In this paper, we derive a mathematical model of a road heating system using underground power distribution lines based on the nonlinear ODE model [1]. A validity of the model is demonstrated via a numerical simulation.

2. Modeling of Road Heating System

2.1. System Settings

In this section, we describe the overall setup of the road heating system using underground power distribution lines studied in this paper. Throughout this paper, we use the term "cable" if we consider the heat of the lines. We show an overview of the system in Fig. 1. In Fig. 1, the yellow and red arrows represent power consumption and supply and thermal distribution, respectively. Electrical power is supplied from the underground power distribution lines to the load electrical equipment (lamp, electric vehicles, etc.) and from the photovoltaic power generation equipment to the underground power distribution lines. In addition, the heat generated in the cables is transferred within the cables and diffused into the ground. Then, the heat propagating to the ground surface is used to melt snow. In this paper, the system is decomposed into four layers: voltage and current distribution in the underground distribution line, thermal dissipation in the cable, thermal diffusion in the ground, and snow melting on the ground surface.

As shown in Fig. 1, the position variable on the underground distribution line is defined as $x\,[\mathrm{m}] \in \mathbb{R}$, with $x = 0$ [m] and $x = L_{\mathrm{f}}$ [m] at the transformer and the end of the underground distribution line, respectively. We also introduce the vertical position variable $y\,[\mathrm{m}] \in \mathbb{R}$ which describes the depth from the ground surface to the cable. We suppose that $y = 0$ [m] for the surface and $y = D_{\mathrm{f}}$ [m], $D_{\mathrm{f}} > 0$ for the outer sheath of the cable.

2.2. Voltage and Current Distribution of Underground Distribution Line

2.2.1. *Setting of underground distribution line*

We give a setting for the underground power distribution line in this paper. We consider the case where the line extends in a straight line from the transformer to the load electrical equipment and photovoltaic generation equipment. In

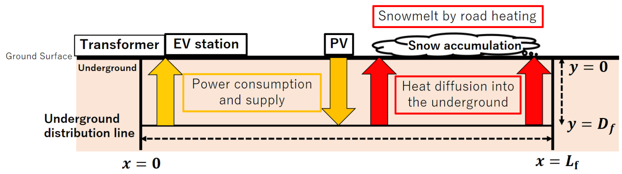


Figure 1: Road heating system using underground power distribution line.

this paper, the underground distribution line refers to a low-voltage underground distribution line. The load electrical equipment is supposed to consume power from the underground distribution line, and the photovoltaic power generation is assumed to supply power to the line. To simplify the discussion, we assume that there are no voltage control devices such as transformers on the intermediate positions over the lines. In this case, the voltage and current on the line varies continuously with the position and time.

2.2.2. *Voltage and current distribution*

The spatial variation on the voltage distribution is described by the nonlinear ODE model [1], which is the following ODEs (1)-(4):

$$\frac{\mathrm{d}\theta(x,t)}{\partial x} = -\frac{s(x,t)}{v(x,t)^2}, \tag{1}$$

$$\frac{\mathrm{d}v(x,t)}{\mathrm{d}x} = w(x,t), \tag{2}$$

$$\frac{\mathrm{d}s(x,t)}{\mathrm{d}x} = \frac{b(x)p(x,t) - g(x)q(x,t)}{g(x)^2 + b(x)^2}, \tag{3}$$

$$\frac{\mathrm{d}w(x,t)}{\mathrm{d}x} = \frac{s(x,t)^2}{v(x,t)^3} - \frac{g(x)p(x,t) + b(x)q(x,t)}{\left(g(x)^2 + b(x)^2\right)v(x,t)}, \tag{4}$$

where $\theta(x,t)\,[\mathrm{rad}] \in \mathbb{R}$, $v(x,t)\,[\mathrm{V}] \in \mathbb{R}$, $s(x,t)\,[\mathrm{V}^2/\mathrm{km}] \in \mathbb{R}$, and $w(x,t)\,[\mathrm{V}/\mathrm{km}] \in \mathbb{R}$ represent the voltage phase, voltage amplitude, supplemental variable, and voltage gradient at the position x and time t, respectively. In (3) and (4), $p(x,t)\,[\mathrm{W}/\mathrm{km}] \in \mathbb{R}$ and $q(x,t)\,[\mathrm{Var}/\mathrm{km}] \in \mathbb{R}$ are active and reactive power consumptions at the position x and time t of the underground distribution line, respectively. Moreover, $g(x)\,[\mathrm{S}/\mathrm{km}] \in \mathbb{R}$ and $b(x)\,[\mathrm{S}/\mathrm{km}] \in \mathbb{R}$ are the conductance and susceptance at the position x of the line.

From Ohm's law, the complex-valued current $\dot{I}(x,t)\,[\mathrm{A}] \in \mathbb{C}$ at the position x and time t is described by the phasor representation

$$\dot{I}(x,t) = \frac{v(x,t)e^{j\theta(x,t)} - v_0 e^{j\theta_0}}{\dot{Z}}, \tag{5}$$

where $\theta_0\,[\mathrm{rad}] \in \mathbb{R}$ and $v_0\,[\mathrm{V}] \in \mathbb{R}$ denote the voltage phase and amplitude of the transformer, respectively.

2.3. Temperature Distribution of Cable

We explain the structure of the single-core XPLE cable [2] employed throughout this paper. For the sake of simplicity, we suppose that a single-phase AC circuit is used to distribute power using the cable. The cross section of a single-core XPLE cable is shown in Fig. 2. The conductor is in the center, covered by the insulation and outer protective sheath, in that order.

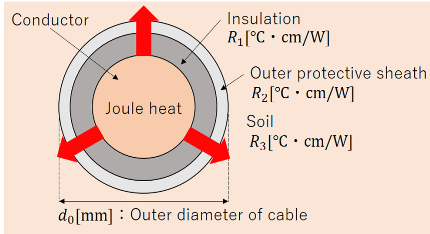


Figure 2: The cross section of single core XPLE cable [2].

In this subsection, we describe the thermal conduction of Joule heat in the conductor to the surroundings, and study the temperature distribution in the cable based on a thermal circuit. The thermal resistances of insulator, outer protective sheath, and soil are denoted by $R_1\,[^\circ\mathrm{C}\cdot\mathrm{cm}/\mathrm{W}] \in \mathbb{R}$, $R_2\,[^\circ\mathrm{C}\cdot\mathrm{cm}/\mathrm{W}] \in \mathbb{R}$, and $R_3\,[^\circ\mathrm{C}\cdot\mathrm{cm}/\mathrm{W}] \in \mathbb{R}$, respectively. Moreover, let $d_0\,[\mathrm{mm}] \in \mathbb{R}$ be the outer diameter of the cable. The potential difference in the electrical circuit corresponds to the temperature difference in the thermal circuit. Moreover, the current in the electrical circuit corresponds to the Joule heat in the thermal circuit. Thus, this heat can be described in terms of the current flowing in the underground distribution line and the effective AC conductor resistance $r(x,t)\,[\Omega/\mathrm{km}] \in \mathbb{R}$. From Ohm's law in thermal circuits, the cable outer surface temperature $\delta_{\mathrm{surf}}(x,t)\,[^\circ\mathrm{C}] \in \mathbb{R}$ at the position x and time t is given by

$$\delta_{\mathrm{surf}}(x,t) = \delta_{\mathrm{base}} + R_3 r(x,t) I(x,t)^2,$$

where $\delta_{\mathrm{base}}\,[^\circ\mathrm{C}] \in \mathbb{R}$ the ambient temperature of the cable.

2.4. Thermal Diffusion in Underground

In this subsection, we derive a mathematical model of underground thermal diffusion is given by the thermal conduction equation. In this paper, we assume that there is no horizontal thermal diffusion, and that the diffusion occurs only in the vertical direction in the underground. We define the underground temperature of the soil at the position x, depth y and time t by $\delta_{\mathrm{soil}}(x,y,t)\,[^\circ\mathrm{C}] \in \mathbb{R}$. Then, we have the thermal conduction equation by the following partial differential equation (PDE):

$$\frac{\partial \delta_{\mathrm{soil}}(x,y,t)}{\partial t} = \frac{\lambda_{\mathrm{diff}}}{D_{\mathrm{f}} - y}\frac{\partial \delta_{\mathrm{soil}}(x,y,t)}{\partial y} + \lambda_{\mathrm{diff}}\frac{\partial^2 \delta_{\mathrm{soil}}(x,y,t)}{\partial y^2}. \tag{6}$$

In (6), $\lambda_{\mathrm{diff}}\,[\mathrm{m}^2/\mathrm{s}] \in \mathbb{R}$ is the thermal diffusivity of soil defined by $\lambda_{\mathrm{diff}} := \frac{\lambda_{\mathrm{tran}}}{c\rho}$, where $c\,[\mathrm{J}/\mathrm{K}\cdot\mathrm{kg}] \in \mathbb{R}$, $\rho\,[\mathrm{kg}/\mathrm{m}^3] \in \mathbb{R}$ and $\lambda_{\mathrm{tran}}\,[\mathrm{W}/\mathrm{m}\cdot\mathrm{K}] \in \mathbb{R}$ are the specific heat, density and thermal conductivity of the soil, respectively.

In order to obtain a solution to the equation (6), the boundary condition at the ground surface and at cable sheath is given by the thermal exchange equation

$$\frac{\partial \delta_{\mathrm{soil}}(x,0,t)}{\partial t} = \lambda_{\mathrm{diff}} \frac{\partial^2 \delta_{\mathrm{soil}}(x,0,t)}{\partial y^2} - f_{\mathrm{ground}}(\delta_{\mathrm{soil}}(x,0,t) - \delta_{\mathrm{ground}}(x,t)),$$

where $\delta_{\mathrm{ground}}(x,t)$ [°C] $\in \mathbb{R}$ is the temperature of the air between the ground surface and the snow accumulation, f_{ground} [W/m$^2 \cdot$ K] $\in \mathbb{R}$ is the thermal transfer coefficient.

Based on the cable sheath temperature and thermal exchange, the boundary condition at the sheath is given by the following PDE:

$$\frac{\partial \delta_{\mathrm{soil}}(x,D_{\mathrm{f}},t)}{\partial t} = \lambda_{\mathrm{diff}} \frac{\partial^2 \delta_{\mathrm{soil}}(x,D_{\mathrm{f}},t)}{\partial y^2} + f_{\mathrm{surf}}(\delta_{\mathrm{surf}}(x,t) - \delta_{\mathrm{soil}}(x,D_{\mathrm{f}},t)),$$

where f_{surf} [W/m$^2 \cdot$ K] $\in \mathbb{R}$ denotes the thermal transfer coefficient in the cable and underground.

2.5. Snow melting on ground surface

In the subsection, we give an equation which describes the spatial and temporal variation of snow melting. We assume that there is no snowfall in order to clarify the amount of snow melted by the road heating system.

The temporal variation of snow volume is represented as an interaction with the sunlight and Joule heat of cables by

$$\frac{\mathrm{d}h_{\mathrm{snow}}(x,t)}{\mathrm{d}t} = -\frac{a_{\mathrm{snow}}}{10 d_{\mathrm{snow}}}(\mu_1 + \mu_2(x,t)), \tag{7}$$

where d_{snow} [g/cm^3] $\in \mathbb{R}$ is the density of snow volume, and $a_{\mathrm{snow}} \in \mathbb{R}$ is the unit conversion factor from [W/m^2] to [mm/s]. Moreover, μ_1 [W/m^2] $\in \mathbb{R}$ is the constant which represents the snowmelt due to sunlight, and $\mu_2(x,t)$ [W/m^2] $\in \mathbb{R}$ represents the snowmelt due to Joule heat of the cable. The snowmelt due to the influence of sunlight is given by $\mu_1 = f_{\mathrm{r}} + f_{\mathrm{s}} + f_{\mathrm{l}}$, where f_{r} [W/m^2] $\in \mathbb{R}$, f_{s} [W/m^2] $\in \mathbb{R}$, and f_{l} [W/m^2] $\in \mathbb{R}$ represent net radiation, sensible heat flux, and latent heat flux, respectively. The incidence on the snow volume is taken as a positive value, and the heat radiation is taken as a negative value. In this paper, the detailed derivation of the sensible and latent heat fluxes are omitted due to a space limitation. By Newton's law, the snowmelt due to Joule heat of the cable is expressed as $\mu_2(x,t) = c_1(\delta_{\mathrm{soil}}(x,0,t) - \delta_{\mathrm{snow}})$, where c_1 [W/m$^2 \cdot$ K] $\in \mathbb{R}$ represents the heat transfer coefficient between the ground and snow.

3. Numerical Validation

In this section, we verify a validity of the mathematical model of the road heating system derived in this paper based on a numerical simulation.

3.1. Simulation setting

We consider the distribution system which is shown in Fig. 3. At first, the reference values for the p.u. values are summarized in Table 1, which are used in the simulation of the voltage distribution of underground distribution lines. Next, the p.u. values used to simulate the voltage distribution of underground distribution lines are shown in Table 2. The parameters related to thermal diffusion are shown in Table 3. Note that the depth at which the heat source of a typical road heating system is buried is 10 cm. In this paper, in order to verify the effectiveness of underground power distribution lines as a heat source, the depth of the cable was also set to 10 cm to fit that value.

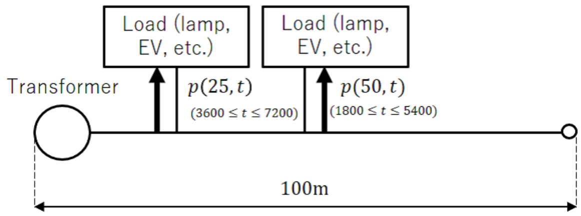


Figure 3: Distribution system considered in this simulation.

Table 1: Reference values of the voltage distribution [1][3]

Parameter	Symbol	Value
Nominal voltage	V_{base}	200 [V]
Length of underground power distribution line	L_{base}	100 [m]
Impedance	Z_{base}	0.0166 [Ω]
Conductance	G_{base}	7.576 [km/Ω]
Susceptance	B_{base}	9.901 [km/Ω]

Table 2: p.u. values of the voltage distribution [1]

Parameter	Symbol	p.u. value
Transformer voltage amplitude	v_0	1
Length of underground power distribution line	L_{f}	1
Conductance	$g(x)$	1
Susceptance	$b(x)$	1

In the following, we describe the simulation conditions of the road heating system considered in this section. We perform a simulation that corresponds to 7200 [s] (2 hours) in real time. It is assumed that when the active power consumption is negative, the load electrical equipment consumes the power in the underground power distribution line. Throughout this section, the absolute value of the active power consumption is set to a large value in order to vary the simulation. The reactive power consumption is assumed to be 0 [p.u.]. We suppose that electric loads such as lamps and electric vehicles are installed at horizontal positions 25 [m] and 50 [m]. We consider the active power

Table 3: Parameters related to thermal diffusion and snow melting.

Parameter	Symbol	Value
Ambient temperature of the cable	δ_{base}	5 [°C]
Cable burial depth	D_{f}	10 [cm]
Density of snow volume	d_{snow}	0.4 [g/cm^3]
Unit conversion factor	a_{snow}	2.986×10^{-6}
Initial condition for snow volume	$h_{\mathrm{snow}}(x, 0)$	3 [cm]
Snow temperature	δ_{snow}	0 [°C]
Net radiation	f_{r}	20 [W/m^2]
Sensible heat flux	f_{s}	4.7×10^{-3} [W/m^2]
Latent heat flux	f_{l}	1.1847×10^{-4} [W/m^2]
Heat transfer coefficient	c_1	280 [W/m$^2 \cdot$ K]

consumptions $p(25, t)$ and $p(50, t)$ given as follows:

$$p(25, t) = \begin{cases} 0 & (t \in [0, 3600)), \\ -5 & (t \in [3600, 7200]), \end{cases}$$

$$p(50, t) = \begin{cases} 0 & (t \in [0, 1800) \cup (5400, 7200]), \\ -5 & (t \in [1800, 5400]). \end{cases}$$

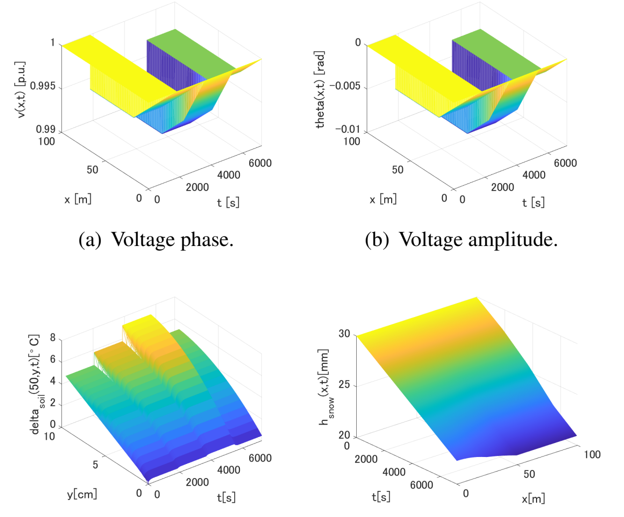
3.2. Simulation results

In this subsection, we perform a numerical simulation given for the settings in the previous subsection to demonstrate the validity of the mathematical model of the proposed road heating system.

From Fig. 4 (a), we see the variations of the voltage phases at t = 1800, 3600, 5400 [s] when the active powers changes due to the electrical equipment in the load. We can also confirm that the voltage amplitude is reduced by the power consumption of the load from Fig. 4 (b).

Next, we show the temperature distribution in the ground relative to the vertical direction at horizontal position x = 50 [m] in Fig. 4 (c). From this figure, it can be confirmed that the ground temperature decreased from the outer sheath of the cable toward the ground surface. Furthermore, the temperature change became smaller as the cable closes to the ground surface.

Finally, Fig. 4 (d) shows the variation of the snow volume with respect to time and position. We can see that a reduction of the snow volume as time goes on. It can also be confirmed that there is a difference in the volume depending on the horizontal position. The above series of numerical observations verify the validity of the mathematical model of the proposed road heating system. In particular, after the cable heats up, it takes approximately one minute for the snowmelt to begin. This indicates that the heat takes one minute to propagate through the underground.



(a) Voltage phase.
(b) Voltage amplitude.
(c) Temperature distribution in the ground at x = 50 [m]
(d) Snow volume

Figure 4: Simulation results

4. Conclusions

In this paper, we have shown a road heating system using underground power distribution lines. We have derived a mathematical model of the system as a main result. The validity of the model was verified based on the numerical simulation.

Acknowledgments

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