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Obtaining networks from time series: what to do with symbolic time series

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Abstract—There has been much recent interest in contructing networks from time series data, partly because doing so makes available a vast new suite of tools for nonlinear time series analysis. In this paper we introduce an alternative approach by which these techniques can be employed when the underlying time series is a discrete symbolic sequence with a small finite alphabet — obvious connections to Markov chains and nonlinear time series modelling will be discussed. A trivial extension to this method also means that it can be applied to continuous-valued scalar time series as well.

1. Introduction

Since the work of Zhang and Small [9, 10] there has been an increasing interest in representing time series data as a complex network. Currently, there are three popular approaches: the *visibility graph* method [2], the *phase space network* [8] and the *recurrence network* [4]. The basic premise of all of these methods is to somehow construct a transformation which represents a time series as a complex network. The hope is that, as a consequence of a clever construction, the properties of the complex network are somehow related to the dynamics of the underlying system,

The difference between each of these methods is in how they achieve the same basic objective. Recurrence and phase space networks represent each individual state (observed at some instance in time) as a node and add links between sufficiently similar (close in phase space) states. This approach is closest to the original cycle network method proposed in [10]: in that paper, each cycle of a time series is a node and links are drawn between similar nodes. All of these methods rely on the principle of proximity: dynamical states are mapped to nodes, and links drawn between nodes corresponding to states which are sufficiently close.

The weakness of these *proximity* methods is that they do not directly encode the dynamical behaviour of the original system. Of course, the adjacency matrix of the corresponding complex networks do resemble recurrence matrices [3] — particularly for the so-called recurrence networks. However, the temporal ordering of rows and columns of a network's adjacency matrix are ignored by all network based measures. Since proximity-based methods rely on reconstructing the dynamical system state from a (usually) scalar time series, it is possible to encode dynamical information through an appropriate over-embedding of the system state: by using an embedding dimension $d \gg d_e$ where d_e is the *"correct"* dimension required for reconstruction, additional information could be included in each state. These excessively large embedding dimensions mean that each state is essentially a strand from a trajectory of the original system. However, both recurrence networks and phase space networks are constructed under the assumption of a"proper" embedding. Hence, these methods can, in principle at least, be considered as constructing networks which represent primarily the topological properties of the attractor underlying the original dynamical system

Conversely, the visibility graph method *does* directly encode dynamical information. Each scalar observable is mapped to a node and links drawn between successive nodes based on a convexity (and implicitly self-simillar) criterion. The visibility graph method, by construction is well suited to stochastic processes, however it is less clear that it is equally applicable to arbitrary dynamical systems.

In this paper we focus on an alternative type of network construction algorithm which will also lead to a *dynamical* network. The methods has been applied (intermittently, and in particular contexts) in the past (see, for example, several examples listed in the review [1]). However, here we pursue a more systematic study of this method for two reasons: (1) as a method to deal with symbolic timeseries, and (2) as a way to explicitly study dynamical properties of the original system — rather than topological properties of the attractor.

2. State transition networks

Let s_t be a discrete symbolic sequence where $s_t \in \mathcal{A}$ a finite alphabet. Alternatively, one may start with x_t a scalar time series and first quantise x_t into b equally probable bins $\mathcal{A} = \{[X^{(0)}, X^{(1)}], [X^{(1)}, X^{(2)}] \ldots [X^{(b-1)}, X^{(b)}]\}$ such that $\text{Prob}(x_t \in [X^{(i-1)}, X^{(i)}] = \frac{1}{b}$ for $i = 1, \ldots b$. Choose an embedding dimension d (as usual, we assume that some convenient prescription can be found to obtain an appropriate dimension [5]) and construct the embedded quantised state $z_t = (s_t, s_{t-1}, \ldots, s_{t-d+1})$.

We note that

$$z_t \in \underbrace{\mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}}_{d \text{ times}}$$

and therefore, in the case of an underlying scalar time series, z_t is defined by the particular d dimensional hypercube of intervals

$$[X^{(\ell_1-1)}, X^{(\ell_1)}] \times [X^{(\ell_2-1)}, X^{(\ell_2)}] \times \ldots \times [X^{(\ell_d-1)}, X^{(\ell_d)}]$$

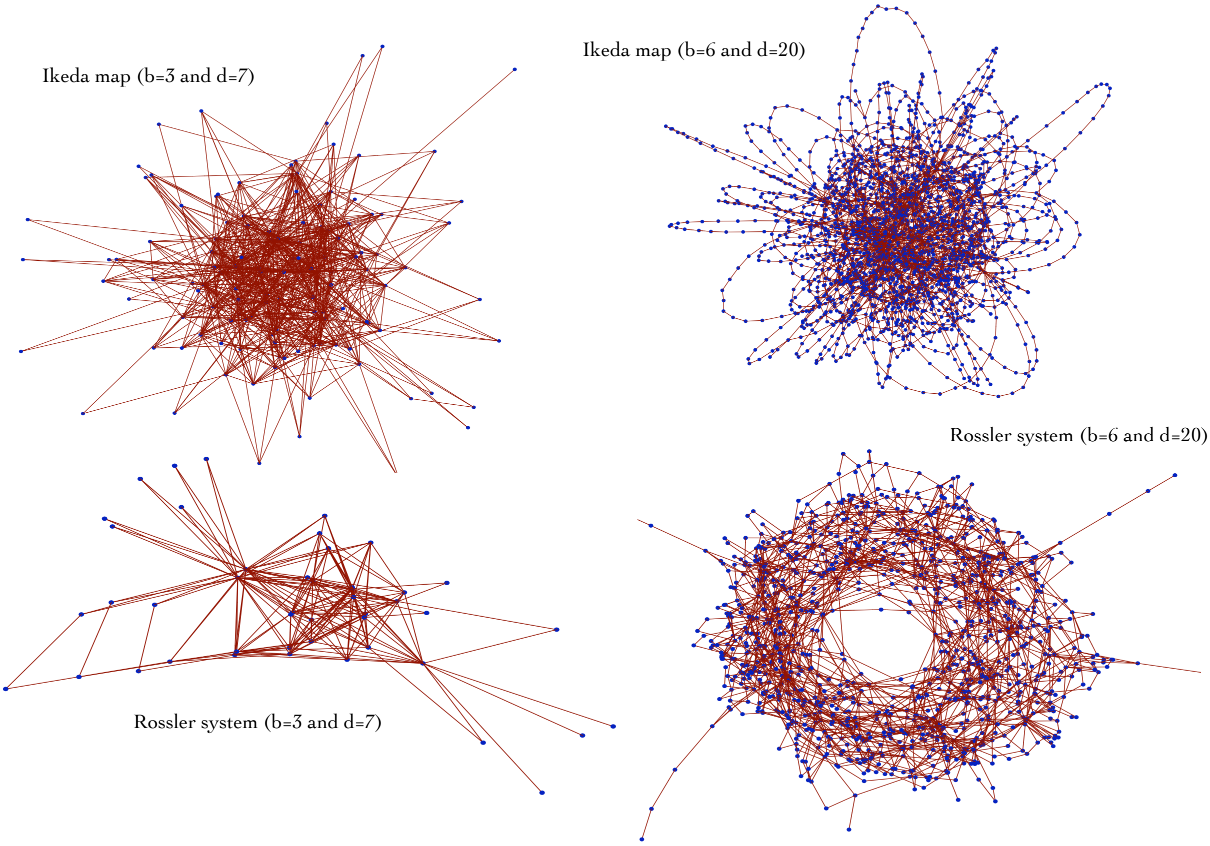

Figure 1: Networks for the chaotic Ikeda map (upper) and the chaotic Rössler system (lower) are depicted. These networks are constructed with quantisation b and embedding d. On the left are networks with a relative coarse quantisation ($b = 3$ and $d = 7$) and on the right a much finer representation ($b = 6$ and $d = 20$). In both cases, the length of the original time series $N = 3162$.

where $\ell_i \in \{1, 2, \ldots d\}$ for $i = 1, 2, \ldots d$. Hence, a discrete embedded state z_t is equivalent to a d-tuple of symbols from an alphabet of b symbols, and also a d-dimensional hypercube.

Finally, to construct a network we let each discrete state z_t correspond to a node $n(t)$. Two nodes $n(i)$ and $n(j)$ are linked if there exists t such that $z_t = z_i$ and $z_{t+d} = z_j$. By definition the link from $n(i)$ to $n(j)$ is directed (although the measures we employ in this preliminary report do not depend on that directedness). The time shift d between z_t and z_{t+d} is included to ensure that the two nodes $n(i)$ and $n(j)$ linked only through completely independent states. This ensures d degrees of freedom among the potential neighbours of $n(i)$ and a (potentially) fully populated adjacency matrix. Conversely, if we had chosen to link $n(i)$ and $n(j)$ on the basis of z_t and z_{t+1} for some t, then there would only be a maximum of b possible links, since $d-1$ of the states of z_{t+1} are determined by z_t. Note also that the size of the network does not depend directly on the length of the time series N (although it does depend on some combination of N, ergodicity and largest Lyapunov exponent), but it depends rather on the alphabet size b and the embedding dimension d: the network size is bounded above by b^d. Illustrative networks are shown in Fig. 1 and will be discussed in the next section.

Clearly, this network is now measuring something quite different from the proximity based methods: connectivity between nodes indicates dynamical causality between coarse grained states in the underlying system. Hence, both coverage of phase space (and hence allowable states) and dynamical behaviour will directly influence the behaviour of these networks. That is, in the case where s_t is obtained by quantising a scalar time series x_t, this method will measure the intersection between the attractor and coarse grained hyper-cubes: a rudimentary proxy for the correlation integral. For the case where we are only concerned with the discrete symbolic sequence, this method constructs a network which encapsulates a Markov chain model of the underlying dynamics.

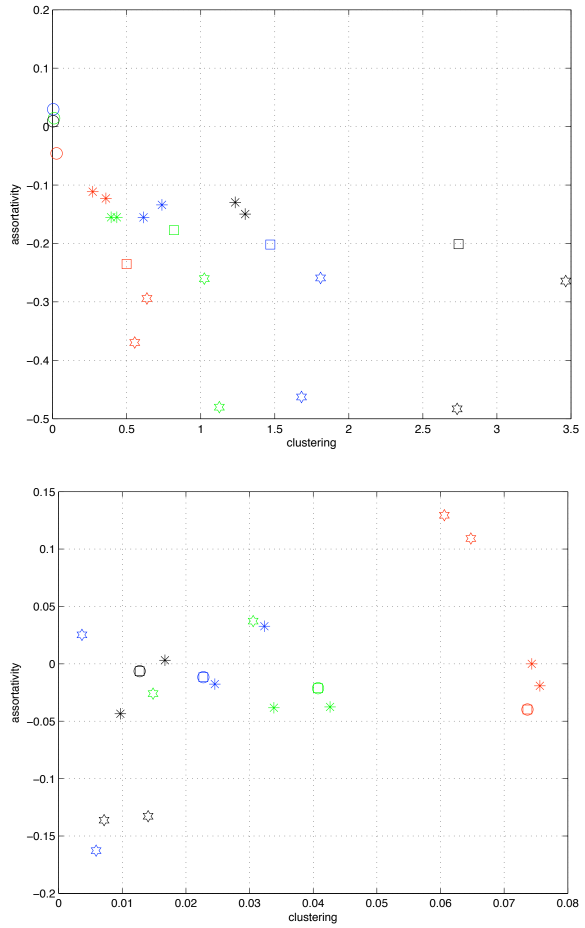

Figure 2: Assortativity and clustering coefficient computed for an array of networks. The upper panel clearly shows we are able to distinguish between iid numbers (circles), linear noise (squares), chaotic maps (asterisks) and chaotic flows (stars) — independent of the length of the signal. The lower panel shows that with increasingly fine quantisation and over-embedding, stochastic systems behave identically (for a fixed length of time series). The clustering coefficient computed here uses a weighted network and therefore is not normalised to a fractional value.

3. Examples

Figure 1 clearly illustrates that the results of this method depend on the choice of parameters d and b. In this figure we only show results for two chaotic systems (which have hence been quantised): the Ikeda map and Rössler chaos. Results for other chaotic systems (not shown here) are similar. For stochastic processes the networks tend to consist of either long isolated strands (since the random system exhibit entirely unique paths which, in a sufficiently high embedding dimension never self-intersect by chance) or a random network (when the sequence of visitation to quantised states is also essentially random). Which of these extremes manifests for random data depends only on the parameters b and d.

With a finer granularity and over-embedding ($b = 6$ and $d = 20$, on the right in Fig. 1) the individual deterministic trajectories become evident — the rate at which splittings in these paths occur is related to the degree of mixing in the underlying system and hence (presumably) the largest Lyapunov exponent. For a coarser quantisation and lower embedding dimension the system transitions between a finite set of discrete possible states — this is particularly evident for the Rössler system (with $b = 3$ and $d = 7$) as the corresponding network is highly structured and symmetric (and this is even more evident for the cases $b = 2$, which are not shown here). A consequence of this fine grained quantisation is that only populated states in phase space will generate corresponding nodes. The network one obtains is therefore a function of the distribution of points in phase space and hence depends on the correlation integral at the corresponding granularity.

In Fig. 2 we show that by simply measuring the network assortativity and clustering coefficient we are able to distinguish between different types of dynamical systems — even for relative short time series. In Fig. 2 networks are constructed from time series of length $N = 568$ (red, colour online), $N = 1000$ (green), $N = 1778$ (blue) and $N = 3162$ (black). The different systems are chaotic maps (Ikeda and logistic maps, shown as asterisks), chaotic flows (Lorenz and Rössler systems, shown as stars) and random signals (independent and identically distributed or *iid* noise and linear noise, shown as circles and squares, respectively). The upper panel is with a relatively coarse embedding ($b = 3$ and $d = 7$), the lower panel is for a finer representation ($b = 6$ and $d = 20$).

Clearly, the coarse embedding is sufficient to distinguish chaos from noise — even for relatively short time series ($N = 568$). The reason is clear from Fig. 1. The network constructed with this method encapsulated both the structured distribution of states in phase space and the deterministic sequence in which these states occur. It turns out that a simple combination of clustering coefficient and assortativity is sufficient to measure this. In contrast, for a finer quantisation and over embedding, the network one obtains from a random signal will always be the same (for a fixed N) — a random sequence of states distributed randomly in state space. Hence, in the lower panel of Fig. 2 we see that the results for the iid and linear noise source exactly coincide (for a fixed N).

In Fig. 3 we briefly explore the effect of time series length N on the median path length of the corresponding network. While, as N increases the network will not get smaller (in terms of number of nodes), it does not necessarily continue to increase in size — particularly when N is sufficiently large to explore all the permitted distinct discrete states. Hence, in Fig. 3 we observe a plateau, where as N increases the network size does not increase, but rather the average degree may increase (up to some bound for de-

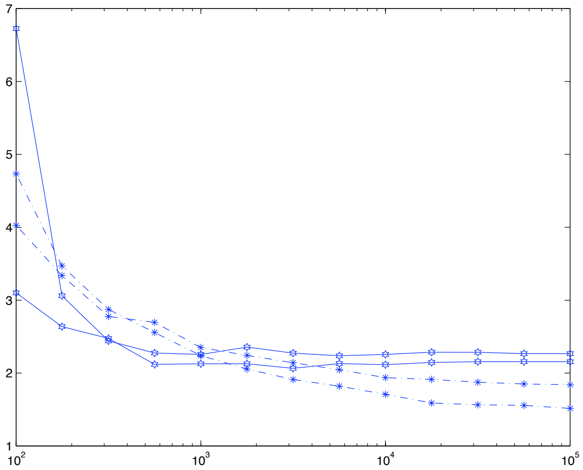

Figure 3: Network diameter as a function of N for chaotic systems. Results for chaotic flows (solid lines) and chaotic maps (dash-dotted) show the average path length is not only not increasing, but decreases to a plateau as a function of N. Note that N is the length of the time series — not the size of the network.

terministic systems) and hence the average path length continues to drop. Of course, for random systems — where increasing N will allow additional quantised states to be visited — this will not be the case.

4. Conclusions

There have been several methods proposed to transform time series into complex networks, and this paper discusses one more. While the method we discuss here is not necessarily original (it has been described implicitly or more-or -less explicitly in several particular contexts), it is important to realise that this approach is in direct contrast to the proximity based methods that are enjoying current popularity. By using a temporal based method, we are able to construct networks for which there is a clear connection to the underlying dynamics — and to the quantities which one usually estimates for nonlinear time series. By exploiting this connection we are able to construct networks which a sensitive characterisation of the features of the underlying dynamical system.

Clearly, we are not arguing that the currently popular proximity methods are inferior to dynamical networks such as those presented here. Only that both styles of networks should be considered. In particular, one area for application of these dynamical networks is in the construction of surrogates. Just as has been done with Twin Surrogates [7] (and the references therein) and pseudo-periodic surrogates [6]: the dynamics are encoded in the networks described here. Be replaying the network evolution, one can construct alternative realisations of the same system. Hence, with this method it is possible to use the network as a archetypal description of the dynamical system underlying the data: the network acts as a model from which distinct realisations of the same dynamics may be obtained.

Acknowledgments

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