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Nonlinear Vibration Property of Local Resonators in Dynamics of an Acoustic Metamaterial

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Abstract—Nonlinear dynamics of local resonators of an acoustic metamaterial is investigated. The fully nonlinear formulation is performed, and the asymptotic analysis of equations of motion is performed in weakly nonlinear dynamics. Both theoretical equations are verified numerically, and it is found a good agreement of them. Then, numerical study is performed in detail in weakly nonlinear dynamics. It is found that DC component and vibrations with local resonance frequency and twice higher frequency than an eigen frequency arise in local resonators. Moreover, this local resonance leads to quasi-periodic oscillation of local resonators.

1. Introduction

In recent years, acoustic metamaterials or artificial periodic structures have attracted great attention of many researchers. Local resonator, which has been introduced by Liu. *et. al.*[1], is often employed as unit structures. Many researchers have confirmed numerically and experimentally that artificial unit structures enable materials to have newly dynamical characteristics such as negative effective mass density[2] and negative effective bulk modulus[3] around frequencies of local resonance. Huang and Sun have proposed a 1D mechanical model with local resonators, which has extreme Young's modulus[4]. Its unit structure consists of two local resonators which act external force on mass points in the oblique direction to that of motion of mass points (i.e., wave propagating). Owing to this geometrical constraint, effects of local resonators on wave propagation becomes anharmonic with respect to wave amplitude.

It is known that nonlinear interactions generally cause complex dynamics; for example, subharmonic/high-harmonic resonance, unstabilization and chaotic behavior, and, especially in the case that the system has discrete structure, intrinsic localized modes[5]. In addition, by using analogy of acoustic metamaterials, research developments of mechanical metamaterials[6] or seismic metamaterials[7] for controlling various types of waves have been accelerated recently. Thus, it is more important to understand nonlinear dynamical property of periodic structures like acoustic metamaterials.

In this study, we construct a mechanical model based on Huang and Sun's concept and focus on the nonlinear dynamics of local resonators by means of numerical simulation and asymptotic analysis.

2. Model

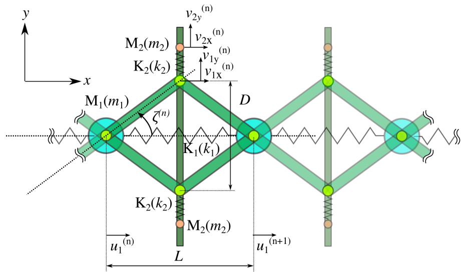


Figure 1: A unit structure.

We consider a periodic structure which is constructed by combining $N(=50)$ unit structures along x-axis, as shown in Figure 1. A unit structure consists of a mass point M_1, an elastic spring K_1, and two "local resonators." A local resonator consists of a mass point M_2 and an elastic spring K_2, and five massless rigid bars. A size of unit structure without external force is L and D in x- and y- direction, respectively. Variables u_1, v_{2i}, and v_{1i} $(i = x, y)$ represent displacements of M_1, M_2, and a massless connecting point which joints rigid bars and K_2, respectively. ζ is the angle between x-axis and a rigid bar. A subscription n of each variables is a unit number. Owing to the symmetry of unit structure, M_1 is allowed to move only in x-direction. M_2 and K_2 are supported by a vertical rigid bar so that motion of M_2 is restricted in the center of unit structure. Therefore, $v_{1x}^{(n)}$ and $v_{2x}^{(n)}$ are represented in terms of $u_1^{(n)}$ and $u_1^{(n+1)}$,

$$v_{1x}^{(n)} = v_{2x}^{(n)} = \frac{u_1^{(n)} + u_1^{(n+1)}}{2}. \tag{1}$$

We ignore the effects of friction by assuming that friction is negligible small. We ignore the effect of gravity.

Equations of motion of M_1 and M_2 are given as Eq. (2) and Eq. (3), respectively,

$$m_1 \frac{d^2 u_1^{(n)}}{dt^2} = k_1(u_1^{(n+1)} - 2u_1^{(n)} + u_1^{(n-1)}) + \frac{k_2(v_{2y}^{(n)} - v_{1y}^{(n)})}{\tan\zeta^{(n)}} - \frac{k_2(v_{2y}^{(n-1)} - v_{1y}^{(n-1)})}{\tan\zeta^{(n-1)}}$$

$$-\frac{m_2}{2}\left(\frac{d^2u_1^{(n-1)}}{dt^2}+2\frac{d^2u_1^{(n)}}{dt^2}+\frac{d^2u_1^{(n+1)}}{dt^2}\right), \quad (2)$$

$$m_2\frac{d^2v_{2y}^{(n)}}{dt^2} = -k_2\left(v_{2y}^{(n)}-v_{1y}^{(n)}\right), \quad (3)$$

where $v_{1y}^{(n)}$ and $\tan\zeta^{(n)}$ are

$$v_{1y}^{(n)} = -\frac{D}{2}+\sqrt{\left(\frac{D}{2}\right)^2 - L\frac{u_1^{(n+1)}-u_1^{(n)}}{2} - \left(\frac{u_1^{(n+1)}-u_1^{(n)}}{2}\right)^2}, \quad (4)$$

$$\tan\zeta^{(n)} = \frac{2v_{1y}^{(n)}+D}{u_1^{(n+1)}-u_1^{(n)}+L}. \quad (5)$$

Eqs. (2) and (3) can be nondimensionalized by introducing nondimensional parameters as follows; mass ratio $\theta = m_2/m_1$, spring constant ratio $\delta = k_2/k_1$, aspect ratio of a unit structure $\mu' = L/D$, and angular frequency ratio $\eta = \omega/\omega_0$, and new time scale $T^* = \omega_0 t$. Here, $\omega_0 = \sqrt{k_2/m_2}$ is the local resonance frequency. In the following, $\bar{(\cdot)}$ represents terms nondimensionalized by L.

By using Taylor expansions of Eqs. (2) and (3) in terms of $u_1^{(n)}$ and $u_1^{(n+1)}$, it's found that nonlinear terms are represented as both of odd- and even-order terms of $u_1^{(n+1)}-u_1^{(n)}$ multiplied by functions of μ'. This indicates that the nonlinearity of mechanical model depends on difference of neighboring displacement of unit structure and its aspect ratio μ'.

By linearizing equations of motions, we obtain the dispersion relation of mechanical model,

$$\{1+\theta(\cos\xi+1)\}\eta^4 + 2\frac{\theta}{\delta}(1-\cos\xi)$$
$$-\left\{1+\theta(\cos\xi+1)+\theta\left(\frac{2}{\delta}+\mu'^2\right)(1-\cos\xi)\right\}\eta^2 = 0, \quad (6)$$

where $\xi = q/L$ and q is a wavenumber. According to Eq. (6), there is a band gap up to $\eta = 1$, i.e., a local resonance frequancy[4].

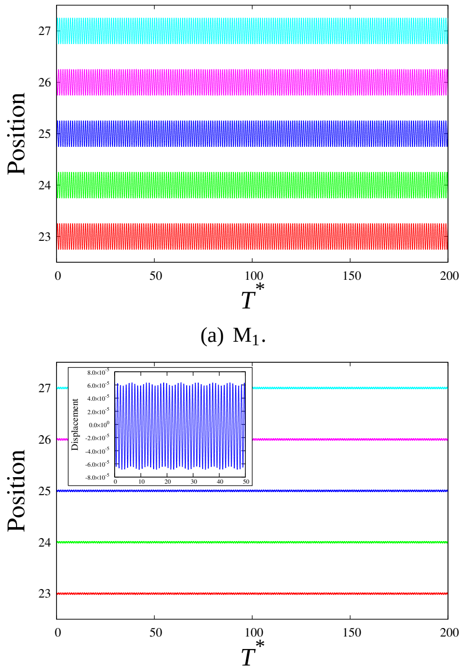
2.1. Numerical calculation and results

In order to investigate the dynamical properties of mechanical model, we perform numerical calculations of free vibration of the system. Nondimentional parameters are selected as $(\theta, \delta, \mu') = (2.0, 0.5, 2.0)$. Numerical integration is performed by the 4th order Runge-Kutta method until $T^* = 200$ and timestep of that is $\Delta T^* = 10^{-4}$. The initial displacement is assigned to M_1 and M_2 as,

$$\bar{u}_1^{(n)}(0) = (-1)^n\bar{A}, \quad \bar{v}_2^{(n)}(0) = \frac{\mu'}{1-\tilde{\eta}^2}\bar{A}, \quad (7)$$

which is one of normal modes with $\xi = \pi$ found in the linearized equations of motion under the periodic boundary condition,$u_1^{(N+1)} \equiv u_1^{(1)}$. Here, $\tilde{\eta}$ is the largest eigen angular frequency derived from Eq. (6) (in this case, $\tilde{\eta} \approx 5.7$).

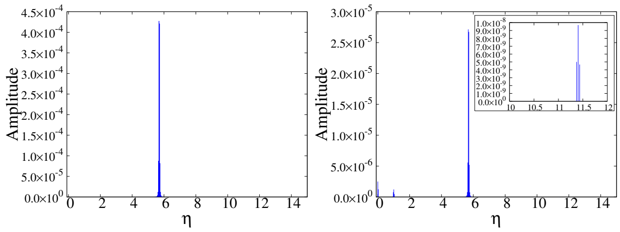
Figure 2 shows the temporal evolution of displacements of M_1 and M_2 at the positions $n = 23, 24, \cdots, 27$, where $\bar{A} = 0.001$ (the inset in Fig.2(b) is temporal evolution of displacement of M_2 at the position $n = 25$ in $T^* = 0-50$). The results of spectral analysis of M_1 and M_2 at the position $n = 25$ in the same case of that in Fig.2 are shown in Fig.3.



(a) M_1.

(b) M_2.

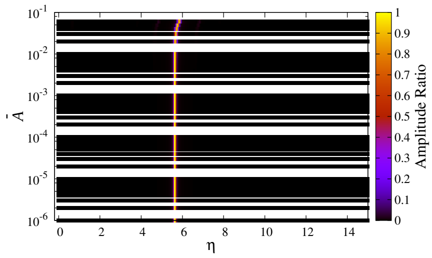
Figure 2: Temporal evolution of displacements($\bar{A} = 0.001$).



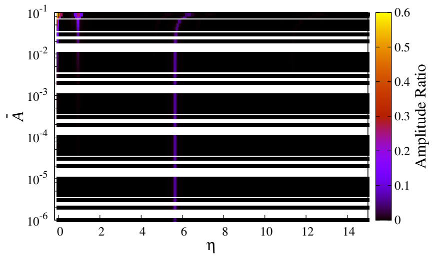
(a) M_1. (b) M_2.

Figure 3: Results of spectral analysis ($\bar{A} = 0.001$).

It is found that as shown in Fig.2(a) and Fig.3(a), M_1 oscillates with constant amplitude and single frequency, on the other hand, as shown in Fig.2(b) and Fig.3(b), the vibration amplitude of M_2 varies and oscillations at $\eta = 0$ (DC component), $\eta \approx 1.0$, and $\eta \approx 2\tilde{\eta}$ are excited. It should be noted that each excited oscillation except the eigen vibration is same phase and amplitude at all position n, and that each center of vibration of M_2 shifts in the direction compressing K_2 (that is, DC component is negative). Then, Fig.4 shows the frequency profile with various eigen oscillation ampliudes $\bar{A}$ under the same conditions of Fig.2.



(a) M_1.



(b) M_2.

Figure 4: Frequency profile vs. $\bar{A}$.

Fig.4 indicates that though large eigen frequency shift cannot be seen in the region of $\bar{A}$ smaller than 10^{-2}, eigen freuquency becomes higher as $\bar{A}$ increases in the region of $\bar{A}$ larger than 10^{-2}. In this region, vibrations of both of M_1 and M_2 are no longer stable.

2.2. Asymptotic analysis

In the region $\bar{A} \leq 10^{-2}$, that corresponds to the weakly nonlinear region, we use the ansatz of displacements of M_1 and M_2,

$$\bar{u}_1^{(n)} = \varepsilon \tilde{u}_1^{(n)} + \varepsilon^2 (\delta u_1^{(n)}) + O(\varepsilon^3), \quad \bar{v}_{2y}^{(n)} = \varepsilon \tilde{v}_{2y}^{(n)} + \varepsilon^2 (\delta \bar{v}_{2y}^{(n)}) + O(\varepsilon^3), \tag{8}$$

where $|\varepsilon| \ll 1$. By substituting Eqs.(8) into Eqs. (2) and (3) and assembling terms at $O(\varepsilon)$ and $O(\varepsilon^2)$ leads to Eqs. (9) and (10),

$$\ddot{\tilde{u}}_1^{(n)} = \left(\frac{\theta}{\delta} + \frac{\theta\mu'}{2}\right)\left(\tilde{u}_1^{(n+1)} - 2\tilde{u}_1^{(n)} + \tilde{u}_1^{(n-1)}\right) + \theta\mu'\left(\tilde{v}_{2y}^{(n)} - \tilde{v}_{2y}^{(n-1)}\right) - \frac{\theta}{2}\left(\ddot{\tilde{u}}_1^{(n-1)} + 2\ddot{\tilde{u}}_1^{(n)} + \ddot{\tilde{u}}_1^{(n+1)}\right), \tag{9}$$

$$\ddot{\tilde{v}}_{2y}^{(n)} = -\tilde{v}_{2y}^{(n)} - \frac{\mu'}{2}\left(\tilde{u}_1^{(n+1)} - \tilde{u}_1^{(n)}\right), \tag{10}$$

and Eqs. (11) and (12),

$$\begin{aligned}\delta\ddot{\bar{u}}_1^{(n)} = {} & \left(\frac{\theta}{\delta} + \frac{\theta\mu'}{2}\right)\left(\delta\bar{u}_1^{(n+1)} - 2\delta\bar{u}_1^{(n)} + \delta\bar{u}_1^{(n-1)}\right) \\ & + \theta\mu'\left(\delta\bar{v}_{2y}^{(n)} - \delta\bar{v}_{2y}^{(n-1)}\right) + \theta\left(\mu'^2 + 1\right) \\ & \times \left\{\tilde{v}_{2y}^{(n)}\left(\tilde{u}_1^{(n+1)} - \tilde{u}_1^{(n)}\right) - \tilde{v}_{2y}^{(n-1)}\left(\tilde{u}_1^{(n)} - \tilde{u}_1^{(n-1)}\right)\right\} \\ & + \frac{3}{4}\theta\left(\mu'^2 + 1\right)\left\{\left(\tilde{u}_1^{(n+1)} - \tilde{u}_1^{(n)}\right)^2 - \left(\tilde{u}_1^{(n)} - \tilde{u}_1^{(n-1)}\right)^2\right\} \\ & - \frac{\theta}{2}\left(\delta\ddot{\bar{u}}_1^{(n-1)} + 2\delta\ddot{\bar{u}}_1^{(n)} + \delta\ddot{\bar{u}}_1^{(n+1)}\right),\end{aligned} \tag{11}$$

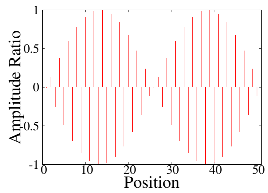
$$\begin{aligned}\delta\ddot{\bar{v}}_{2y}^{(n)} = {} & -\delta\bar{v}_{2y}^{(n)} - \frac{\mu'}{2}\left(\delta\bar{u}_1^{(n+1)} - \delta\bar{u}_1^{(n)}\right) \\ & - \frac{1}{4}\left(\mu'^2 + 1\right)\left(\tilde{u}_1^{(n+1)} - \tilde{u}_1^{(n)}\right)^2,\end{aligned} \tag{12}$$

where $(\ddot{\cdot}) \equiv d^2/dT^{*2}$. Since Eqs. (9) and (10) are identified with linearlized equations of motion, $\tilde{u}_1^{(n)}$ and $\tilde{v}_{2y}^{(n)}$ are obtained as follow,

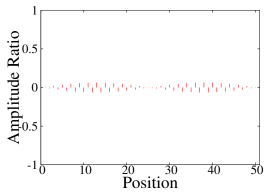
$$\tilde{u}_1^{(n)} = \bar{A}_n \cos(\eta T^*), \quad \tilde{v}_{2y}^{(n)} = \frac{\mu'}{1-\eta^2}\bar{A}_n \cos(\eta T^*), \tag{13}$$

where $\bar{A}_n = \bar{A}\cos(n\xi)$, $\bar{A}$ is constant, and the relationship between ξ and η is represented as Eq. (6).

In order to compare the solution of Eq. (9)-(12) with that of Eqs. (2) and (3), we perform numerical calculations of free vibrations of the system for both formulations. The same numerical calculation method and conditions of Section 2.1 are used, and some eigen scillation patterns are selected as initial displacements. One of the results of temporal evolution at the position $n = 25$ initial displacement pattern are shown in Fig.5 and Fig.6, respectively. Here, $\max(\bar{A}_n) = 0.001$.

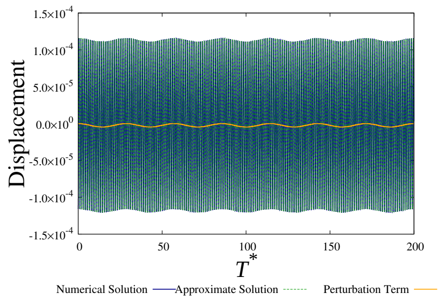


(a) M_1.

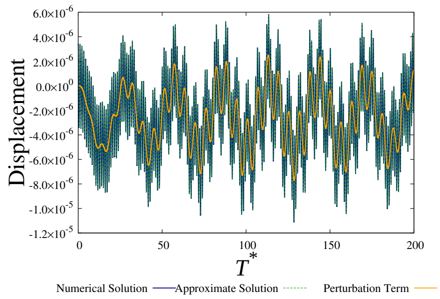


(b) M_2.

Figure 5: Initial displacement pattern.



(a) M_1.



(b) M_2.

Figure 6: Comparison of temporal evolution. (a),(b): The line of apploximate solution (green dashed line) overlaps the line of numerical solution (blue solid line) considerably.

It is found that as shown in Fig.6, the ansatz (in Fig.6, "Approximate Solution") agrees with the solution of Eqs. (2) and (3)(in Fig.6, "Numeical Solution"). The results at other position or these with other initial displacements also agree with the solutions. Therefore, dynamics in weakly nonlinear region can be represented well by the ansatz Eq. (8) and its evolution equations Eqs. (9) and (12).

By substituting $\xi = \pi$ (which is the same normal mode applied to M_1 and M_2 in the case of Fig.2) into Eq. (13), both of the third and forth term on the right-hand side of Eq. (11) are eliminated, which indicates that Eq. (11) is identical with Eq. (9). Thus, by the assumption that $\delta\overline{u}_1^{(n)}(0) = 0$, no perturbation of M_1 is caused in the region $\overline{A} \leq 10^{-2}$, i.e. $\delta\overline{u}_1^{(n)}(T^*) = 0$. Inserting $\delta\overline{u}_1^{(n)}(T^*) = 0$ into Eq. (12) leads to Eq. (14),

$$\delta\dot{\overline{v}}_{2y}^{(n)} = -\delta\overline{v}_{2y}^{(n)} - \frac{1}{2}\left(\mu'^2 + 1\right)\overline{A}^2\left\{1 - \cos(2\eta_l T^*)\right\}, \quad (14)$$

where η_l is one of the frequencies obtained by substituting $\xi = \pi$ into Eq. (6). Therefore, the solution $\left(\delta\overline{v}_{2y}^{(n)}\right)_{\mathrm{sol}}$ is obtained,

$$\begin{aligned}\left(\delta\overline{v}_{2y}^{(n)}\right)_{\mathrm{sol}} \quad = \quad & \frac{(\mu'^2+1)(2\eta_l^2-1)}{4\eta_l^2-1}\overline{A}^2\cos(T^*) \\ & + \frac{\mu'^2+1}{2(4\eta_l^2-1)}\overline{A}^2\cos(2\eta_l T^*) - \frac{1}{2}\left(\mu'^2+1\right)\overline{A}^2.\end{aligned} \quad (15)$$

This result agrees with the results of spectral analysis(Fig.3) which shows that DC component and oscillations at $\eta = 1$ and twice as high frequency as eigen frequency η_l of local resonators are excited. According to Eq. (15), it is found that in the case of $\xi = \pi$, i.e., the eigen oscillation with minimum wavelengths, the amount of DC component (the third term on the right-hand side of Eq. (15)) depends only on aspect ratio μ' whereas both of amplitude of vibration at $\eta = 1$ and that at $\eta = 2\eta_l$ depend on not only μ' but eigen frequency η_l.

It should be noted that in the case of $\xi = \pi$, oscillation at the local resonance frequency ($\eta = 1$) is excited according to Eq. (15). If the same nondimensional parameters are selected, we obtain $\eta_l = (5 \pm \sqrt{41})/2$ (the higher one is $\tilde{\eta}$), which are irrational numbers. Thus, the frequency ratio of η_l to local resonance frequency is irrational so that M_2 vibrates quasi-periodically[8].

3. Conclusion

We constructed a mechanical model based on an acoustic metamaterial consisting of local resonators which interact nonlinearly with propagating waves; and investigated numerically dynamics of free vibration under the periodic boundary condition. The results show that in the region of vibration amplitude $\overline{A}$ smaller than 10^{-2}, only exitation of some vibrations of local resonators occurs, on the other hand, that in the region of $\overline{A}$ larger than 10^{-2}, eigen frequency shift occurs. Then, we performed asymptotic analysis in weakly nonlinear region under the periodic boundary condition, and confirmed that the ansatz can represent dynamics well. Moreover, we found that in the case of $\xi = \pi$, vibrations of local resonators with at local resonance frequency and twice as high one as the eigen frequency, and DC component of which the amount only depends on the aspect ratio μ' are excited because of the nonlinearity. As the ratio of some eigen frequencies of mechanical model to local resonance frequency is irrational, local resonators will vibrate quasi-periodically in case of some normal modes.

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