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Consistency and complexity in coupled semiconductor lasers with time-delayed optical feedback

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Abstract– Consistency of response in a system driven repeatedly by a complex signal has been observed in many nonlinear dynamical systems. We investigate the consistency of unidirectionally coupled semiconductor lasers with optical feedback and measure the complexity of the coupled laser system by using Lyapunov spectrum. It is found that the complexity of the coupled laser system can be classified into three regions. When the system has consistency, the complexity of the coupled laser system corresponds to that of the Drive laser. In inconsistency regions, the complexity of the coupled laser system corresponds to the sum of those of the uncoupled Drive and Response lasers. The complexity increases in the boundary of the consistency region at negative optical frequency detuning. The complexity strongly depends on the degree of consistency.

1. Introduction

Many nonlinear dynamical systems have an ability to generate consistent outputs when driven by a repeated external signal, and this phenomenon is referred to as consistency [1]. The concept of consistency is illustrated in Fig. 1. We consider a situation where a nonlinear dynamical system (called a response system) is driven by a repeated complex signal such as a chaotic or noise signal. The response system may not produce similar temporal outputs because of different initial conditions for different trials of drive input. However, if the response system has consistency, an identical complex temporal waveform of the response system can be obtained at each repetition of the drive input. Consistency can be defined as the ability of a dynamical system to produce an identical response output after some transient period, when the system is driven by a repeated drive signal.

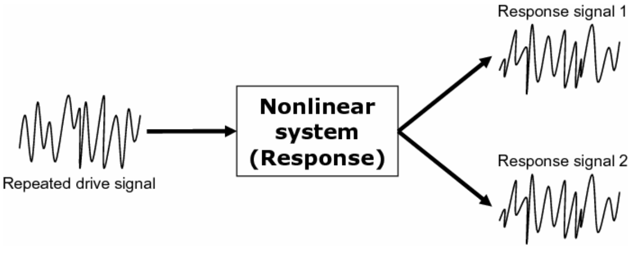

Fig. 1 Concept of consistency.

Consistency of response has been experimentally observed in a solid-state laser [1]. The concept of consistency could be applied for an implementation of physical one-way function [2], where an output signal can be easily produced from an input signal through a complex function, whereas the input signal cannot be estimated from the output signal. The physical implementation of one-way function has been reported with a token with complex speckle scattering patterns of light [2]. Instead of using spatial complex patterns, temporal dynamics may be useful when a dynamical system has consistent response. The use of consistency may lead to a new technique of the implementation of physical one-way function, which could be a key technique for hardware-oriented information security systems.

One of the important characteristics of physical one-way function is the complexity of the functional system. The complexity of a single semiconductor laser with time-delayed optical feedback has been reported [3]. However, the complexity of optically coupled semiconductor lasers has not been reported. Moreover, the relationship between consistency and complexity in coupled lasers has not been investigated yet. It is important to investigate how complexity changes when the state of consistency is changed in coupled laser systems.

In this study we investigate consistency of unidirectionally coupled semiconductor lasers with time-delayed optical feedback and measure the complexity of the coupled laser system by using Lyapunov spectrum. We quantitatively evaluate the complexity of the coupled laser system by using an entropy and dimensionality, which are estimated from the Lyapunov spectrum.

2. Model for numerical simulations

We numerically investigate the conditions to obtain consistency in unidirectional coupled semiconductor lasers with time-delayed optical feedback. A model consisting of three semiconductor lasers (called Drive, Response 1 and Response 2 lasers) is shown in Fig. 2. Consistency is the ability to generate similar temporal outputs when a response system is driven by a repeated external signal. Instead of using a repeated drive signal, we prepare the Response 2 laser which is a copy of the Response 1 laser with the same parameter values and different initial conditions. Both the Response 1 and 2 lasers are subject to a common chaotic signal from the Drive laser. We consider that the Response laser has consistency when similar temporal waveforms are obtained between the Response 1 and 2 lasers. All the three lasers are subject to time-delayed optical feedback to induce chaotic intensity fluctuations.

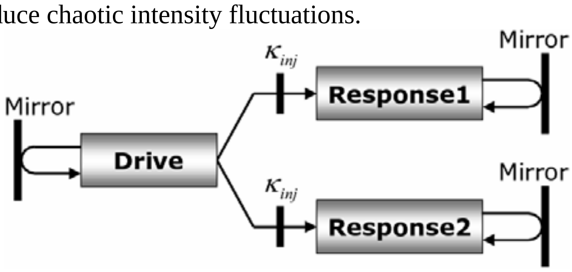

Fig. 2 Model.

The model shown in Fig. 2 can be described by a set of coupled rate equations for semiconductor lasers, which are known as the Lang-Kobayashi equations [4,5]. The unidirectionally coupled Lang-Kobayashi equations are described as follows,

Drive laser:

$$\dot{E}_D(t) = \frac{1+i\alpha}{2}\left\{G_N\left(N_D(t)-N_0\right)-\frac{1}{\tau_p}\right\}E_D(t) + \kappa_D E_D(t-\tau)\exp(-i\omega_D\tau) \quad (1)$$

$$\dot{N}_D(t) = J_D - \frac{N_D(t)}{\tau_s} - G_N\left(N_D(t)-N_0\right)\left|E_D(t)\right|^2 \quad (2)$$

Response 1 laser:

$$\dot{E}_{R1}(t) = \frac{1+i\alpha}{2}\left\{G_N\left(N_{R1}(t)-N_0\right)-\frac{1}{\tau_p}\right\}E_{R1}(t) + \kappa_R E_{R1}(t-\tau)\exp(-i\omega_R\tau) + \kappa_{inj}E_D(t-\tau_{inj})\exp\left(i(\Delta\omega t-\omega\tau_{inj})\right) \quad (3)$$

$$\dot{N}_{R1}(t) = J_R - \frac{N_{R1}(t)}{\tau_s} - G_N\left(N_{R1}(t)-N_0\right)\left|E_{R1}(t)\right|^2 \quad (4)$$

Response 2 laser:

$$\dot{E}_{R2}(t) = \frac{1+i\alpha}{2}\left\{G_N\left(N_{R2}(t)-N_0\right)-\frac{1}{\tau_p}\right\}E_{R2}(t) + \kappa_R E_{R2}(t-\tau)\exp(-i\omega_R\tau) + \kappa_{inj}E_D(t-\tau_{inj})\exp\left(i(\Delta\omega t-\omega\tau_{inj})\right) \quad (5)$$

$$\dot{N}_{R2}(t) = J_R - \frac{N_{R2}(t)}{\tau_s} - G_N\left(N_{R2}(t)-N_0\right)\left|E_{R2}(t)\right|^2 \quad (6)$$

where E is the complex electric field and N is the carrier density. The subscripts D, $R1$, and $R2$ represent the Drive, Response 1, and Response 2 lasers, respectively. G is the gain coefficient, α is the linewidth enhancement factor, N_0 is the carrier density at transparency, τ_s is the carrier lifetime, τ_p is the photon lifetime, τ is the round-trip delay time in the external cavity, and τ_{inj} is the propagation time of the injection light from the Drive to Response lasers. The feedback coefficient κ is given by $\kappa = \left(1-r_2^2\right)r_3/\left(r_2\tau_{in}\right)$, where τ_{in} is the round-trip time in the internal laser cavity. r_2 and r_3 represent intensity reflectivities of the laser facet and the external mirror. $\omega_D = 2\pi c/\lambda_D$ and $\omega_R = 2\pi c/\lambda_R$ are the angular frequency of the solitary Drive and Response lasers, where λ_D and λ_R are the optical wavelengths of the solitary Drive and Response lasers. The coupling strength from the Drive to the two Response lasers is given by the injection coefficient κ_{inj} . $\Delta\omega = \omega_D - \omega_R$ is the optical angular frequency detuning between the solitary Drive and Response lasers, and $\Delta f = \Delta\omega/2\pi$ represents the optical frequency detuning between the solitary Drive and Response lasers.

We numerically solve Eqs. (1) ~ (6) by using the fourth-order Runge-Kutta method. We set the same parameter values for the Response 1 and Response 2 lasers, whereas different parameter values are used between the Drive and Response lasers. All the three lasers start from different initial conditions. Consistency is observed when identical temporal outputs of the Response 1 and 2 lasers are obtained.

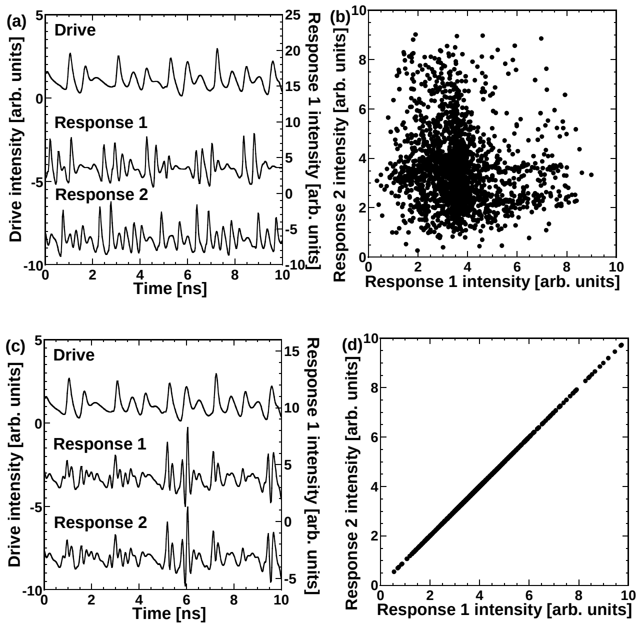

Fig. 3 (a),(c) Temporal waveforms of the three lasers, and (b),(d) correlation plots between the Response 1 and 2 lasers. The coupling strength is (a),(b) $\kappa_{inj} = 0.0$, and (c),(d) $\kappa_{inj} = 0.24$ ns^{-1}.

3. Consistency in coupled semiconductor lasers

Figure 3(a) and 3(b) show the temporal waveforms of the three semiconductor lasers and the corresponding correlation plot between the Response 1 and 2 lasers without optical injection from the Drive laser. It is found that the outputs of the Response 1 and 2 lasers show different temporal behaviors, and the Response lasers do not show consistency. Next the output of the Drive laser is injected into the two Response lasers. Figures 3(c) and 3(d) show the temporal waveforms of the three lasers and the corresponding correlation plot between the Response 1 and 2 lasers at the coupling strength of $\kappa_{inj} = 0.24$ ns^{-1}. The temporal waveforms of the Response 1 and 2 in are identical, even though they are different from the temporal waveform of the Drive laser, as shown in Fig. 3(c). The correlation plot shown in Fig. 3(d) is a straight line at 45 degree, indicating the achievement of consistency of the Response lasers. Consistency of the Response laser outputs can be achieved by increasing the coupling strength κ_{inj}.

We quantitatively evaluate the degree of consistency by using the cross-correlation function as follows,

$$C = \frac{\left\langle \left(I_{R1}(t) - \bar{I}_{R1}\right)\left(I_{R2}(t) - \bar{I}_{R2}\right)\right\rangle}{\sigma_{R1}\sigma_{R2}}, \tag{7}$$

where $I(t)$ is the intensity of the Response lasers, $\bar{I}$ is the mean value of the laser intensity, σ is the standard deviation of the laser intensity, and $<\cdot>$ is the time average. The subscripts $R1$ and $R2$ represent the Response 1 and Response 2 lasers, respectively.

Figure 4 shows the two dimensional map of the cross-correlation value C as functions of the coupling strength κ_{inj} and the optical frequency detuning Δf. The degree of consistency is represented by using gray scale, and the black region corresponds to high consistency ($C \approx 1$). It is found that consistency is obtained in the region of large coupling strengths κ_{inj} and slightly negative detunings Δf. This region almost corresponds to the injection locking range, where the optical wavelengths are matched between the Drive and Response lasers by optical injection [5].

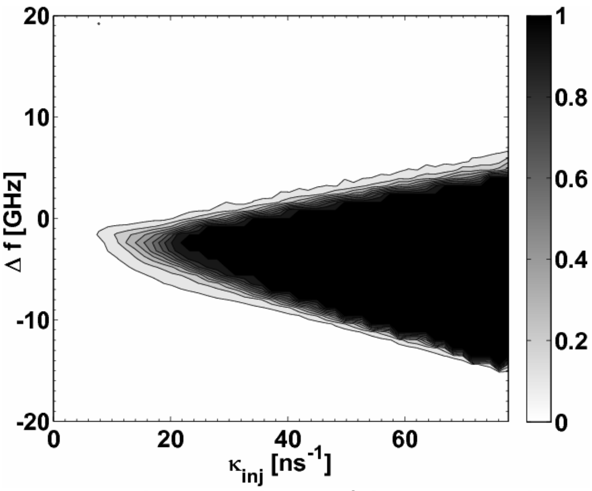

Fig. 4 Two dimensional map of the cross correlation C between the Response 1 and 2 lasers as functions of the coupling strength κ_{inj} and optical frequency detuning Δf. Black region corresponds to high consistency

4. Lyapunov spectrum analysis

To calculate the complexity of the coupled laser system, we calculate Lyapunov exponents. We derive linearized equations for small deviations from the original rate equations of Eqs. (1) ~ (6). We numerically solve the linearized equations, and calculate a norm of the linearized variables. For time-delayed nonlinear dynamical systems, all the linearized variables that are included in the delay time need to be regarded as independent variables for the calculation of the norm [3,6]. The maximum Lyapunov exponent can be calculated from time average of the logarithm of the norm.

For multi-dimensional nonlinear dynamical systems, a number of Lyapunov exponents exist, which are called Lyapunov spectrum. It is necessary to use a number of sets of the linearized equations to calculate Lyapunov spectrum for time-delayed dynamical systems. Each Lyapunov exponent can be obtained from each set of the linearized variables by using the orthogonalization of the vector of the linearized variables [3].

Kolmogorov-Sinai entropy (KS entropy, h_{KS}), which represents unpredictability of dynamical systems, can be obtained from the Lyapunov spectrum. The upper limit of KS entropy can be calculated from the sum of positive Lyapunov exponents [3],

$$h_{KS} = \sum_{i|\lambda_i > 0} \lambda_i \tag{8}$$

KS entropy indicates a loss rate of information. A large value of KS entropy indicates that the system has large unpredictability.

Kaplan-Yorke dimension (KY dimension, D_{KY}, also known as Lyapunov dimension), which represents dimensionality of dynamical systems, can be also estimated from the Lyapunov spectrum. KY dimension can be calculated as follows [3],

$$D_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{\lambda_{i+1}}, \tag{9}$$

where j satisfies the following relationship,

$$\sum_{i=1}^{j} \lambda_i > 0 > \sum_{i=1}^{j+1} \lambda_i. \tag{10}$$

KY dimension indicates the number of variables to represent dynamical systems. A large number of KY dimension corresponds to more complex dynamics of the systems.

Figure 5(a) shows the two-dimensional map of KS entropy h_{KS} for the coupled semiconductor laser system as functions of the coupling strength κ_{inj} and the optical frequency detuning Δf. Black regions correspond to large KS entropy. It is found that the complexity of the coupled semiconductor laser system can be classified into three regions. First, KS entropy becomes low (the white region of Fig. 5(a)), which almost corresponds to the region where the Response laser has consistency (the black region of Fig. 4). It is worth noting that the KS entropy in this region corresponds to the KS entropy of the uncoupled Drive laser ($h_{KS} = 0.86$ ns^{-1}). Secondly, when the Response laser does not show consistency, KS entropy becomes larger (the gray region of Fig. 5(a)). The KS entropy in this region corresponds to the sum of the solitary Drive and Response lasers without optical coupling ($h_{KS} = 2.03$ ns^{-1}). More interestingly, KS entropy becomes the largest in the region near the boundary of the consistency (the black region of Fig. 5(a)). The KS entropy in this region is $h_{KS} = 4.37$ ns^{-1} which is larger than the sum of the KS entropy of the solitary Drive and Response lasers. The region of the largest KS entropy only appears in the boundary of the negative optical frequency detuning. This asymmetry may result from the linewidth enhancement factor α in semiconductor lasers [5].

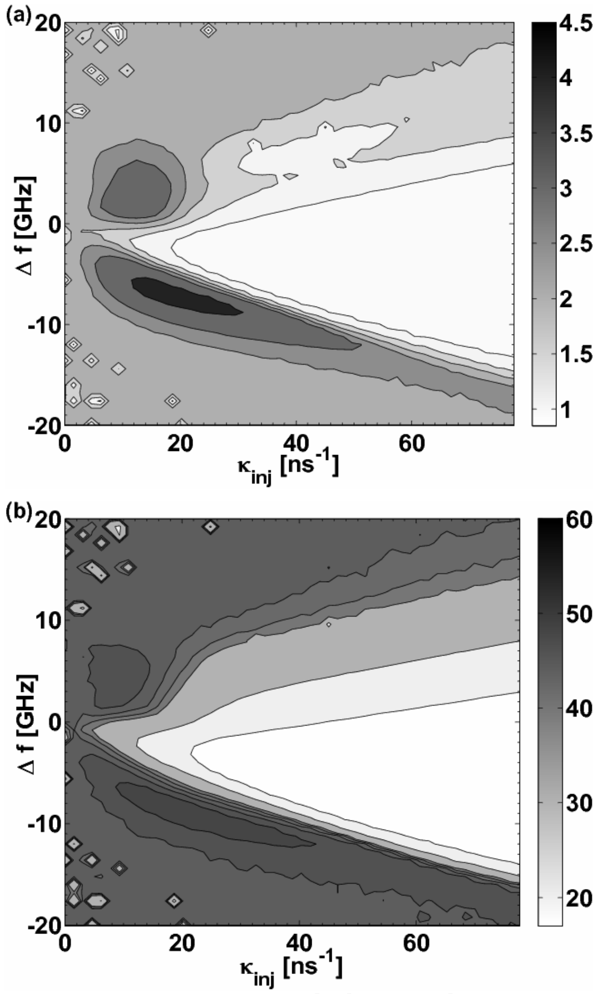

Fig. 5 (a) KS entropy and (b) KY dimension as functions of the coupling strength κ_{inj} and the optical detuning frequency Δf .

Figure 5(b) shows the two-dimensional map of KY dimension D_{KY} for the coupled semiconductor laser system as functions of the coupling strength κ_{inj} and the optical frequency detuning Δf . The three regions of KY dimension can also be found, as in Fig. 5(a). The smallest KY dimension is obtained in the white region of Fig. 5(b), corresponding to the consistency region (the black region of Fig. 4). In this region KY dimension corresponds to that for the uncoupled Drive laser ($D_{KY} = 16.3$). In the inconsistency region, KY dimension becomes larger (the dark gray region of Fig. 5(b)). In this region KY dimension corresponds to the sum of those for the solitary Drive and Response lasers without coupling ($D_{KY} = 44.0$). KY dimension increases slightly in the boundary of the consistency region with the negative optical frequency detuning ($D_{KY} = 49.2$).

5. Conclusion

We have investigated the consistency of unidirectionally coupled semiconductor lasers with optical feedback and measured the complexity of the coupled laser system by using Lyapunov spectrum. We have found that the complexity of the coupled laser system can be classified into three regions. When the system has consistency, the complexity of the coupled laser system becomes small and corresponds to that of the solitary Drive laser. In the inconsistency region, the complexity of the coupled laser system corresponds to the sum of those of the solitary Drive and Response lasers. The complexity increases further in the boundary of the consistency region at the negative optical frequency detuning. The complexity of the coupled laser system strongly depends on the degree of consistency.

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