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Controller Design for 2-Dimensional Nonlinear Control Systems Generating Limit Cycles and Its Application to Spacerobots

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Abstract—In this paper, we propose a controller generating a limit cycle for a 2-dimensional nonlinear control system. First, we state our problem setting and some assumptions. Next, we derive a controller that generates a desired stable limit cycle for the system, and investigate some characteristics of the controller. We then apply our results to control of a spacerobot with an initial angular momentum. Finally, some simulations are carried out to demonstrate effectiveness of the proposed method.

1. Introduction

In various research fields, the concept of limit cycles is quite important. For instance, stable walking or gait of humanoid robots in robot engineering, oscillator circuits in electronic engineering, catalytic hypercycles in chemistry, circadian rhythm in biology, boom-bust cycles in economics and so on. Phenomenon of limit cycles is specific for nonlinear systems, along with chaos, and it has been attracted a lot of researcher's interest. Hence, researches on limit cycles have been vigorously done from mathematical and engineering perspectives so far.

In control theory, a lot of researchers have focused on synthesis problems of systems that generate limit cycles [1]. For example, in recent work, synthesis methods of nonlinear/hybrid systems whose solution trajectories converge to desired limit cycles are proposed in [2, 3, 4], and robust generation of oscillations for a class of nonlinear systems is studied in [5, 6]. On the other hand, the number of studies on design of only control inputs that realize a desired limit cycle for a given nonlinear control system is too small. Since we design only control inputs to make the solution trajectories converge to the desired limit cycle, it seem to be quite difficult to solve the above synthesis problem. However, such a synthesis method is desperately-needed in not only nonlinear control theory, but also applications.

In this paper, we develop a controller design method that generates limit cycles for 2-dimensional nonlinear control systems. We first state our problem setting and give some assumptions. Then, we introduce a switching-type controller and derive some nonlinear control properties of the proposed controller. Finally, we treat a planar spacerobot with an initial angular momentum as an engineering example in order to show the availability of the proposed control strategy.

2. Problem Setting

In this paper, we consider a 2-dimensional manifold Q and the following 2-dimensional nonlinear control system defined in an open subset $D \subset Q$:

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x \in D$ is a state, $u \in \mathbf{R}$ is a control input and $f,\ g : D \to TQ$ are smooth vector fields defined in D. We also set *a limit cycle function* $V : D \to \mathbf{R}$, which determines a desired limit cycle in D, and use notations for $V(x)$:

$$\begin{aligned} D_0 &:= \{\, x \in D \mid V(x) = 0 \,\}, \\ D_+ &:= \{\, x \in D \mid V(x) > 0 \,\}, \\ D_- &:= \{\, x \in D \mid V(x) < 0 \,\}. \end{aligned} \tag{2}$$

Thus, the desired limit cycle is D_0. We now give assumptions on $V(x)$ as follows.

Assumption 1: The limit cycle function $V(x)$ satisfies the following properties.

(a) $V(x)$ is a smooth function.

(b) $V(x) = 0$ determines an unique closed curve D_0 in D.

(c) $V(x)$ satisfies

$$VL_fV < 0 \tag{3}$$

at any point such that $L_gV = 0$, where L_fV means the Lie derivative of V along f. ∎

This paper deals with the following problem on generation of a limit cycle for the system (1).

Problem 1: For the 2-dimensional nonlinear control system (1), find a control strategy that makes the desired closed curve D_0 an unique stable limit cycle in D. ∎

3. Proposed Control Strategy

This section gives a solution of Problem 1, that is, a control strategy that makes a solution trajectory of the system (1) a desired limit cycle. We propose the control input:

$$u = \begin{cases} -\dfrac{L_fV + \dfrac{V}{|V|}\sqrt{(L_fV)^2 + L_gV \cdot q(L_gV)}}{L_gV} & (L_gV \neq 0), \\ 0 & (L_gV = 0). \end{cases} \tag{4}$$

In (4), the function $q : D \to \mathbf{R}$ satisfies $q(0) = 0$ and

$$L_g V \cdot q(L_g V) > 0, \tag{5}$$

at any point such that $L_g V \neq 0$. (4) is a kind of switching controller, however, the following guarantees smoothness of (4) when $L_g V = 0$.

Proposition 1: The control input (4) is real-analytic in $D_+ \cup D_-$. ∎

From Propostion 1, we can rewrite (4) as

$$u = -\frac{L_f V + \frac{V}{|V|}\sqrt{(L_f V)^2 + L_g V \cdot q(L_g V)}}{L_g V} \tag{6}$$

without using the cases. Next, we discuss the continuity of (6) at D_0. For $\delta > 0$, set two subsets of D as

$$\begin{aligned} D_{\delta+} &:= \{\, x \in D_+ \mid \|x - y\| < \delta,\ {}^{\exists} y \in D_0 \,\}, \\ D_{\delta-} &:= \{\, x \in D_- \mid \|x - y\| < \delta,\ {}^{\exists} y \in D_0 \,\}, \end{aligned} \tag{7}$$

where $\|\cdot\|$ means the normal norm defined on D. By using (7), we define the small input property of the limit cycle function.

Definition 1: For each $\epsilon > 0$, there exists a $\delta > 0$ such that $|u| < \epsilon$ makes

$$V L_f V + V L_g V u < 0 \tag{8}$$

for $x \in D_{\delta+} \cup D_{\delta-}$. Then, the limit cycle function V is called to satisfy *the small control property*. ∎

We now consider the control input (6) with $q(L_g V) = V^2 (L_g V)^3$, that is,

$$u = -\frac{L_f V + \frac{V}{|V|}\sqrt{(L_f V)^2 + V^2 (L_g V)^4}}{L_g V}. \tag{9}$$

Under the small input property of the limit cycle function, we can prove the next proposition for (9).

Proposition 2: Assume that V satisfies the small input property. Then, the control input (9) is also continuous in D_0. ∎

Applying the control input (9) to the system (1), we can derive the following main theorem on convergence of a solution trajectory of (1).

Theorem 1: By the control input (9), A solution trajectory of (1), which starts an arbitrary initial point $x_0 \in D$, satisfies

$$\lim_{t \to \infty} V(x(t)) = 0, \tag{10}$$

that is, the solution trajectory converges to D_0 as time goes to infinity. ∎

It is noted that Theorem 1 guarantees only convergence of the trajectory to D_0, and does not guarantees the existence of a limit cycle. Denote the closed loop of the system (1) with the controller (9) as

$$\dot{x} = f(x) - \frac{L_f V + \frac{V}{|V|}\sqrt{(L_f V)^2 + L_g V \cdot q(L_g V)}}{L_g V} g(x) =: f_c(x). \tag{11}$$

In order to make the solution trajectory a desired limit cycle, we need to choose a suitable desired limit cycle, that is, V. (i) the vector field f_c of the closed loop system (11) does not have any equilibria around D_0, (ii) the vector field f_c is not parallel to $\nabla V(x)$ around D_0.

Remark 1: If D_0 is equal to the origin, then the control input (6) is coincident with the Sontag-type controller with the control Lyapunov function [7, 8]. In a sense, the proposed controller (6) is an extension of the Sontag-type controller to limit cycles. ∎

4. Application to Spacerobot with Initial Momentum

In this section, we apply the control strategy derived in Section 3 to an physical example. We treat the planar 2-link spacerobot depicted in Fig. 1 [9, 10]. The spacerobot consists of a body and an arm, and they are connected by a joint. Let us denote the absolute angle of the body by $\theta \in [0, 2\pi) =: \mathbf{S}$ and the relative angle of the arm from the body by $\phi \in \mathbf{S}$. For parameters, we set the distance between the joint and the center of the body's (the arm's) centroid as L (l), the mass of the body (the arm) as M (m), the inertia moment of the body (the arm) with respect to its centroid as J_M (J_m). The spacerobot is assumed to have an initial angular momentum $P_0 > 0$.

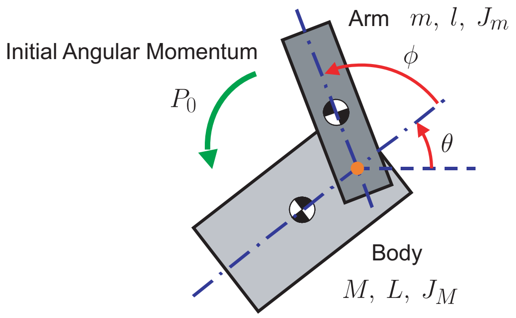

Fig. 1 : Spacerobot with Initial Angular Momentum

The conservation law of angular momentum for the spacerobot is represented by

$$(M_1 + A_1 \cos\phi)\,\dot{\theta} + (M_2 + A_2 \cos\phi)\,\dot{\phi} = P_0, \tag{12}$$

where

$$\begin{aligned} M_1 &:= J_M + J_m + \frac{Mm(L^2 + l^2)}{M + m}, \quad A_1 := \frac{2MmLl}{M + m}, \\ M_2 &:= J_m + \frac{Mml^2}{M + m}, \quad A_2 := \frac{MmLl}{M + m}, \end{aligned} \tag{13}$$

and we assume that $M_1 > A_1$ and $M_2 > A_2$ for the parameters of the spacerobot.

Denote the state variable as $x = [\,\theta \;\; \phi\,]^{\mathsf{T}} \in \mathbf{S} \times \mathbf{S}$, and hence the configuration manifold of the spacerobot is the 2-dimensional torus $\mathbf{T}^2 \approx \mathbf{S} \times \mathbf{S}$. We here assume that the velocity of the arm $\dot{\phi}$ can be controlled, that is $u = \dot{\phi}$. Consequently, from (12), we obtain the nonlinear control system for the spacerobot:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{P_0}{M_1 + A_1 \cos\phi} \\ 0 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} -\dfrac{M_2 + A_2 \cos\phi}{M_1 + A_1 \cos\phi} \\ 1 \end{bmatrix}}_{g(x)} u. \quad (14)$$

It must be noted that the control system (14) has a drift term $f(x)$ arising from the initial momentum. Since $f(x) \neq 0,\ \forall x \in \mathbf{T}^2$, (14) does not have any equilibria, and hence the spacerobot cannot get still.

In [9, 10], the authors derive an analytical time optimal control solution for a 2-link planar acrobot, that is, a control strategy that makes the state of the robot a desired one at a given terminal time. On the other hand, we deal with another control problem. Our aim is to make the body of the spacerobot face to a desired direction. To fulfill this, we set

$$V(x) = \theta - \theta_d, \quad (15)$$

where θ_d is the desired angle of the body. From (14) and (15), we obtain

$$L_f V = \frac{P_0}{M_1 + A_1 \cos\phi},\ L_g V = -\frac{M_2 + A_2 \cos\phi}{M_1 + A_1 \cos\phi}. \quad (16)$$

Since $M_1 > A_1$ and $M_2 > A_2$, we can see that $L_g V \neq 0,\ \forall x \in \mathbf{T}^2$, and then Assumption 1 is satisfied.

Now, we show simulations of control of the spacerobot. Set the parameters as $M_1 = 4$, $A_1 = 2$, $M_2 = 3$, $A_2 = 1$, $P_0 = 2$, and the desired angle of the body as $\theta_d = 0$. The initial state of the spacerobot is given by $x_0 = [\,\pi \;\, \pi/4\,]$. We use the control input (9) with (15) and (16) to achieve our aim.

Fig. 2 shows the time series of θ and ϕ. Note that θ and ϕ take values in the rang of $[0, 2\pi)$. In Fig. 3, the solution trajectory from the initial point on the 2-dimensional torus $\mathbf{T}^2$ is illustrated. From these figures, it can be confirmed that the body's angle θ converges to the desired angle $\theta_d = 0$, and the solution trajectory behaves as a limit cycle. We can verify that for any initial point, the solution trajectory converges to the desired limit cycle.

Fig. 4 depicts the time series of the control input (9). It can be checked that the amplitude of the controller converges to 0 as the value of V tends to 0, then Proposition 2 is satisfied. However, since we simulate the system with numerical computation and the controller switches at the boundary of $\theta = 0$, the chattering phenomenon occurs. The time series of the value of V is shown in Fig. 5. We can see that the value of V converges to 0 as time goes by, and it can be confirmed that Theorem 1 holds.

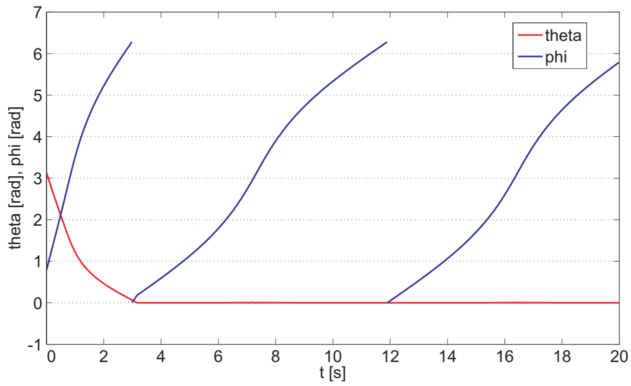

Fig. 2 : Time series of θ and ϕ

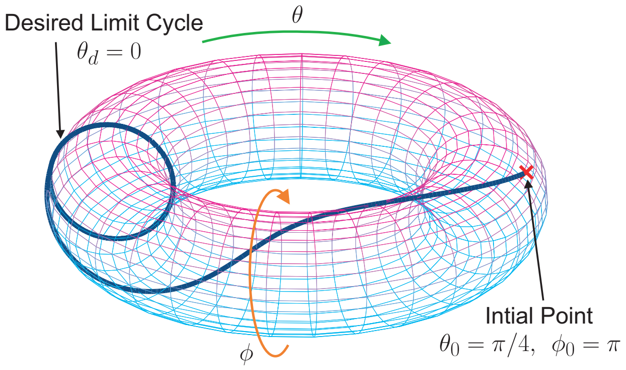

Fig. 3 : Solution Trajectory on Torus $\mathbf{T}^2$

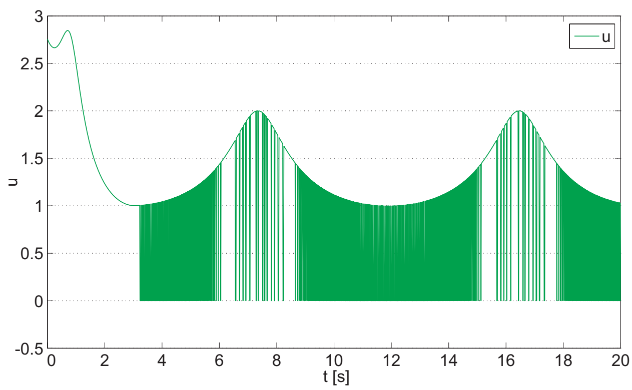

Fig. 4 : Time series of u

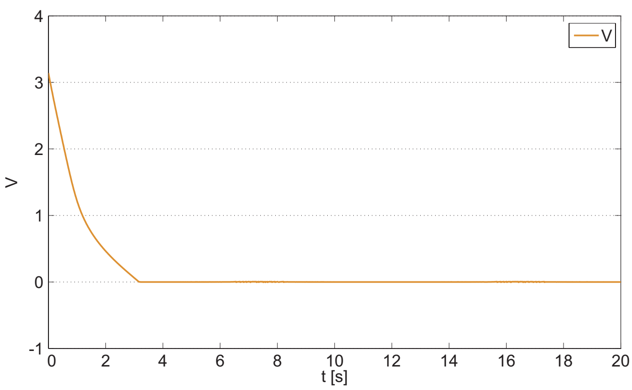

Fig. 5 : Time series of V

We consider an attempt to eliminate the chattering phenomenon as seen in Fig. 4. The chattering phenomenon is caused by the switching function in (9):

$$\frac{V}{|V|} = \begin{cases} 1 & (V > 0), \\ -1 & (V < 0). \end{cases} \tag{17}$$

Now, we introduce a change of (17) in (9) as

$$u = -\frac{L_f V + \frac{V}{|V| + \delta}\sqrt{(L_f V)^2 + V^2 (L_g V)^4}}{L_g V}, \tag{18}$$

where $\delta > 0$ is a constant. We can expect that the controller smoothly switches around the boundary of $V = 0$. Fig. 6 shows the time series of the control input (18) with $\delta = 0.005$. It can be confirmed that the chattering phenomenon does not occur. Moreover, the time series of θ and ϕ is depicted in Fig. 7, and from this we can see that the body's angle θ converges to the desired angle $\theta_d = 0$, and the solution trajectory behaves as a limit cycle, although the controller (9) is modified by (18).

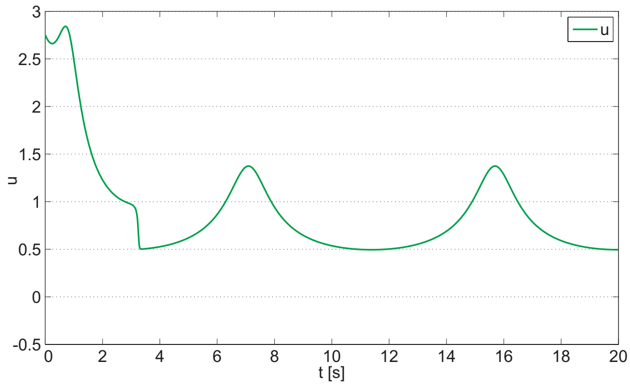

Fig. 6 : Time series of u with (18)

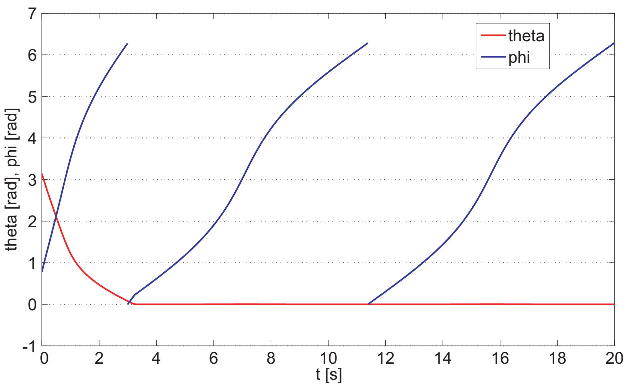

Fig. 7 : Time series of θ and ϕ with (18)

5. Conclusion

We have developed a controller design technique for a 2-dimensional nonlinear control system whose solution trajectory converges to a desired limit cycle. Some characteristics of the proposed controller have been derived from the viewpoint of the smoothness and the small control property. Then, we have applied the proposed control strategy to a control problem of a spacerobot subject to an initial angular momentum. Simulations have shown effectiveness of the proposed control strategy. We consider some future work as follows: (i) controller design for general nonlinear control systems, (ii) applications to physical, biological and engineering examples.

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