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On the Construction of Piecewise Quadratic Lyapunov Functions

Yuzo Ohta† and Hisashi Yokoyama†

†Graduate School of Engineering, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan
Email: tcs.y.ohta@peopole.kobe-u.ac.jp, yokoyama@cs23.cs.kobe-u.ac.jp

Abstract—Piecewise quadratic Lyapunov functions (PQLFs) are proposed by Johansson and Rantzer in 1997. They applied it for stability analysis of piecewise linear systems. It is very effective for some of piecewise linear systems. However, it seems that PQLFs are not so effective for stability analysis of linear uncertain systems or nonlinear uncertain systems with polytopic uncertainties since PQLFs coincide with ordinal QLFs for these systems. In this paper, we propose a method to increase the freedom included in PQLFs for getting less conservative stability results for uncertain systems.

1. Introduction

The Lyapunov Direct Method, which is one of the most powerful methods for stability analysis of nonlinear systems, depends crucially upon our ability to select a function that can establish stability of a given system. For this reason, there has been a large number of papers advancing the computer-based methods for construction of Lyapunov functions. In particular, the Linear Matrix Inequality (LMI) approach was proposed, which is based on Quadratic Lyapunov Functions (QLFs)[1] or Piecewise QLFs (PQLFs)[2], as well as the Linear Programming-Computer Geometry approach which utilizes Polytopic Lyapunov Functions (PLFs) or Piecewise Linear Lyapunov Functions (PLLFs) (see [3] – [6], and references therein). We can reduce conservativeness of robust stability results obtained by using PLFs and PLLFs by increasing freedom included in PLFs and PLLFs. On the other hand, PQLFs are mainly applied for stability analysis of piecewise linear systems and not so powerful in the context of robust stability analysis.

The main objective of this paper is to propose a method to increase the ability of PQLFs in analyzing stability of nonlinear uncertain systems.

Notation. In this paper, $\mathbf{R}$ and $\mathbf{R}^n$ stand for the real number system and the n-dimensional real vector space, respectively. For $x \in \mathbf{R}^n$, $x^\top$ is the transpose of x. For a (finite or infinite) set V, co V denotes the convex hull of V. For a set $V \subseteq \mathbf{R}^n$, int V denotes the set of interior points of V. For sets $V, W \subseteq \mathbf{R}^n$, $V \backslash W$ stands for the complement of W relative to V. For $c \in \mathbf{R}^n$, $\mathcal{H}(c)$ denotes a hyperplane given by $\{x \in \mathbf{R}^n \mid c^\top x = 0\}$. For $x \in \mathbf{R}^n$, $|x| = \sqrt{x^\top x}$. For a polytope $\mathcal{P}$, node $\mathcal{P}$ stands for the set of nodes of $\mathcal{P}$ and $|\text{node } \mathcal{P}|$ stands for the cardinality of node $\mathcal{P}$.

2. Preliminaries

2.1. Systems Description

Let us consider a system given by

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbf{R}^n$ is the system state vector, $f : \mathbf{R}^n \to \mathbf{R}^n$, and $f(0) = 0$.

In this paper, we consider stability of the 0 solution of (1) within a given polytope $X \subseteq \mathbf{R}^n$ such that $0 \in$ int X. Throughout the paper we assume the following:

Assumption 1 *The function f satisfies the generalized sector condition in X, that is, there exist hyperplanes $\{\mathcal{H}(c_\ell), |c_\ell| = 1\}_{\ell=1}^{L_0}$, and piecewise linear functions $\{f_q\}_{q=1}^{Q}$ such that X is decomposed into smaller polytopes $\{X_m\}_{m=1}^{M_0}$ by $\{\mathcal{H}(c_\ell),\ \ell \in \mathcal{L}_0\}$, $\mathcal{L}_0 = \{1, 2, \cdots, L_0\}$ and $f(x)$ satisfies*

$$f(x) \in \text{co } \{f_q(x)\}_{q=1}^{Q}, \quad x \in X \tag{2}$$

$$f_q(0) = 0 \quad \forall q \in Q = \{1, 2..., Q\}, \tag{3}$$

and f_q is given by

$$f_q(x) = A_{m,q}x, \quad x \in X_m, \tag{4}$$

where $m \in \mathcal{M}_0 = \{1, 2, \cdots M_0\}$.

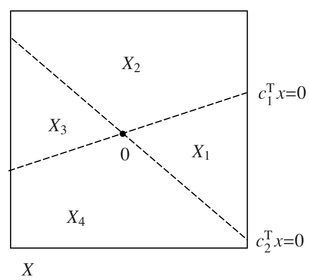

Fig. 1 An example of X, $\{\mathcal{H}(c_\ell),\ \ell \in \mathcal{L}_0\}$, and $\{X_m,\ m \in \mathcal{M}_0\}$, where $L_0 = 2$ and $M_0 = 4$.

Let us consider some additional hyperplanes $\{\mathcal{H}(c_\ell), |c_\ell| = 1\}_{\ell=L_0+1}^{L_0+L_1}$. These hyperplanes divides X_m's, and we suppose that X is divided into I small polytopes $\{\mathcal{X}_i\}_{i=1}^{I}$ such that $X = \cup_{i \in \mathcal{I}} \mathcal{X}_i$, where $\mathcal{I} = \{1, 2, \cdots, I\}$. Note that for each $i \in \mathcal{I}$ there exists a $m \in \mathcal{M}_0$ such that $\mathcal{X}_i \subseteq X_m$.

Given arbitrary $\mathcal{X}_i$, $i \in \mathcal{I}$. Let x_i be an arbitrary interior point of $\mathcal{X}_i$, and define

$$s_{i,\ell} = \begin{cases} 1, & \text{if } \overline{c}_\ell^\top x_i > 0, \\ -1, & \text{if } \overline{c}_\ell^\top x_i < 0, \end{cases} \tag{5}$$

Let $L = L_0 + L_1$ and $\mathcal{L} = \{1, 2, \cdots, L\}$ and

$$E_i = \begin{bmatrix} s_{i,1}c_1^\top \\ s_{i,2}c_2^\top \\ \vdots \\ s_{i,L}c_L^\top \end{bmatrix}. \quad (6)$$

Then, X_i can be represented by

$$X_i = \{x \in X : E_i x \geq 0\}, \quad (7)$$

and we have

$$E_i x = E_j x, \quad x \in (X_i \cap X_j), \quad (8)$$

because $c_\ell^\top x = 0$ for all $\ell \in \mathcal{L}_{i,j}$ and $x \in (X_i \cap X_j)$, where $\mathcal{L}_{i,j} \subseteq \mathcal{L}$ is the set of suffix such that $(X_i \cap X_j) \subseteq \mathcal{H}(c_\ell)$ for all $\ell \in \mathcal{L}_{i,j}$.

2.2. Piecewise Quadratic Lyapunov Functions

The candidate of piecewise quadratic Lyapunov function (PQLF) is given by [2]

$$V(x) = (E_i x)^\top R(E_i x), \quad x \in \mathcal{X}_i, \quad (9)$$

where $R \in \mathbf{R}^{L\times L}$ is a symmetric matrix.

We note that $V(0) = 0$ and $V(x)$ is continuous because of (8). In [2], the following stability result is shown.

Lemma 1 *Let $(\hat{\gamma},\ \hat{\alpha},\ \hat{R},\ \{\hat{U}_i\},\ \{\hat{W}_i\})$ be an optimal solution of*

$$\text{(LMI):} \left\{ \begin{array}{ll} \displaystyle\max_{\gamma,\,\alpha,\,R,\,U_i,\,W_i,\,i\in\mathcal{I}} & \gamma \\ \text{subject to} & (10), (11)\ . \end{array} \right.$$

where

$$\left\{ \begin{array}{l} \alpha I < P_i - E_i{}^\top W_i E_i, \quad P_i = E_i^\top R E_i \\ -\gamma I > A_{m,q}^\top P_i + P_i A_{m,q} + E_i^\top U_i E_i \end{array} \right.$$
$$i \in \mathcal{I},\ \ m:\ \mathcal{X}_i \subseteq X_m,\ \ q \in \mathcal{Q}, \quad (10)$$

$$\alpha > \underline{\alpha},\ \ \gamma < \overline{\gamma},\ \ R = R^\top,\ \ U_i = U_i^\top \geq 0,$$
$$W_i = W_i^\top \geq 0, \quad i \in \mathcal{I}, \quad (11)$$

$\underline{\alpha} > 0$ is a very small constant, $\overline{\gamma} > 0$ is a very large constant, and

$$\overline{A}_{m,q} = \begin{bmatrix} A_{m,q} & b_{m,q} \\ 0 & 0 \end{bmatrix}. \quad (12)$$

If the optimal value $\hat{\gamma} > 0$, then $V(x)$ in (9) is a Lyapunov function for (1), and for any $x_0 \in \Omega(\rho_{\max})$, the solution $x(t; x_0)$ of (1) stays in in $\Omega(\rho_{\max})$ and converges to 0 exponentially, where $\Omega(\rho) = \{x :\ V(x) \leq \rho\}$, and $\rho_{\max} =$ $\max\{\rho > 0 :\ \Omega(\rho) \subseteq X\}$.

2.3. A Motivative Example

The basic idea of PQLF candidate given by (9) is to switch $V(x)$ according to the characteristic of the system at x, that is, $V(x)$ is switched from $(E_i x)^\top R(E_i x)$ to $(E_j x)^\top R(E_j x)$ when a solution moves from X_i to X_j. It seems that robustness issue is not considered so much in [2]. If $M_0 = 1$ and if $L = 0$, then it is just a robust stability issue for linear polytopic uncertain systems, and the condition (10) of Lemma 1 is a sufficient condition for robust stability.

In this paper, we will propose to switch $V(x)$ even in X_i so that we can reduce the conservativeness of stability results. We will do this by considering additional hyperplanes to decompose X_i into smaller polytopes.

Example 1 *Let us consider a system [7] given by*

$$\dot{x} \in \text{co}\,\{A_1 x, A_2 x\}, \quad x \in X = [-7, 7] \times [-7, 7], \quad (13)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.06 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1.94 & -1 \end{bmatrix}. \quad (14)$$

This system is not quadratically stable[7], but by considering additional hyperplanes to decompose X smaller polytopes, we have a PQLF whose level set shown in Fig. 2.

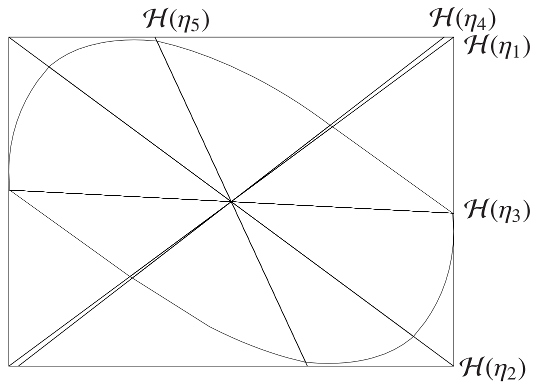

Fig. 2 A Stability region by PQLF

The main issue is how to choose such hyperplanes. In the following sections, we consider this issue.

3. Modification of PQLF Candidates

3.1. Adding a Dividing Hyperplane

When the optimal value $\hat{\gamma}$ of (LMI) is not positive, we increase the freedom included in PQLF candidate $V(x)$ by introducing additional dividing hyperplanes. Let us consider a hyperplane $\mathcal{H}(c_{L'})$, $L' = L + 1$, $|c_{L'}| = 1$, which divides, say J polytopes $\{\mathcal{X}_{i_j}\}_{j=1}^J$ into $\mathcal{X}_{i_j} = \mathcal{X}'_{i_{j_1}} \cup \mathcal{X}'_{i_{j_2}}$, $j =$

$1, 2, \cdots, J$. So far, we have L' hyperplanes $\{\mathcal{H}(c_\ell)\}_{\ell=1}^{L'}$ and X is divided into $I' = I+J$ polytopes $\{\mathcal{X}'_i\}_{i=1}^{I'}$. For simplicity of notation, we suppose that $\mathcal{X}_i = \mathcal{X}'_i$ for $i = 1, 2, \cdots, I-J$, and $\mathcal{X}_i = \mathcal{X}'_i \cup \mathcal{X}'_{i+J}$ for $i = I-J+1, I-J+2, \cdots, I$ (that is, $i_j = i_{j_1} = I - J + j$ and $i_{j_2} = I + j$, $j = 1, 2, \cdots, J$). Let

$$p(i) = \begin{cases} i, & i = 1, 2, \cdots, I, \\ i - J, & i = I+1, I+2, \cdots, I', \end{cases} \tag{15}$$

and we determine $\{s_{i,\ell}\}$ by (5), where $i \in \mathcal{I}' = \mathcal{I} \cup \{I + 1, I+2, \cdots, I'\}$, and $\ell \in \mathcal{L}' = \mathcal{L}_1 \cup \{L+1\}$. We note that $s_{i,\ell} = s_{p(i),\ell}$, for all $i \in \mathcal{I}'$ and $\ell \in \mathcal{L}$.

Define

$$E'_i = \begin{bmatrix} E_{p(i)} \\ s_{i,L'} c_{L'}^\top \end{bmatrix}, \quad R' = \begin{bmatrix} R & r \\ r^\top & \hat{r} \end{bmatrix}, \quad P'_i = E_i'^\top R' E'_i, \tag{16}$$

$$U'_i = \begin{bmatrix} U_i & u_i \\ u_i^\top & \hat{u}_i \end{bmatrix}, \quad W'_i = \begin{bmatrix} W_i & w_i \\ w_i^\top & \hat{w}_i \end{bmatrix}, \tag{17}$$

and consider the corresponding (LMI)

$$(\mathrm{LMI})': \begin{cases} \max\limits_{\gamma', \alpha', R', U'_i, W'_i, i \in \mathcal{I}'} & \gamma' \\ \text{subject to} & (18), (19), \end{cases}$$

where

$$\begin{cases} \alpha' I < P'_i - E_i'^\top W'_i E'_i, \\ -\gamma' I > A_{m,q}^\top P'_i + P'_i A_{m,q} + E_i'^\top U'_i E'_i \end{cases}$$
$$i \in \mathcal{I}', \quad m : \mathcal{X}'_i \subseteq X_m, \quad q \in \mathcal{Q}, \tag{18}$$

$$\alpha' > \underline{\alpha}, \quad \gamma < \overline{\gamma}, \quad R' = R'^\top, \quad U'_i = U_i'^\top \geq 0,$$
$$W'_i = W_i'^\top \geq 0, \quad i \in \mathcal{I}', \quad q \in \mathcal{Q}. \tag{19}$$

Let $(\hat{\gamma}, \ \hat{\alpha}, \ \hat{R}, \ \{\hat{U}_i\}, \ \{\hat{W}_i\})$ be an optimal solution of (LMI). Then, it is easy to see that we have a feasible solution of (LMI)$'$ by setting $R = \hat{R}$, $r = 0$, $\hat{r} = 0$, $U_i = \hat{U}_{p(i)q}$, $u_i = 0$, $\hat{u}_i = 0$, and $W_i = \hat{W}_{p(i)q}$, $w_i = 0$, $\hat{q}_i = 0$ in (16) and(17), $\gamma' = \hat{\gamma}$ and $\alpha' = \hat{\alpha}$.

Therefore, we have the following:

Theorem 1 *Let* $(\hat{\gamma}, \ \hat{\alpha}, \ \hat{R}, \ \{\hat{U}_i\}, \ \{\hat{W}_i\})$ *be an optimal solution of* (LMI). *Then* (LMI)$'$ *has an optimal solution* $(\hat{\gamma}', \ \hat{\alpha}', \ \hat{R}', \ \{\hat{U}'_i\}, \ \{\hat{W}'_i\})$ *such that* $\hat{\gamma}' \geq \hat{\gamma}$.

In the remaining of this section we will consider how to choose a hyperplane $\mathcal{H}(c_{L'})$ such that the optimal value $\hat{\gamma}'$ of (LMI)$'$ satisfies $\hat{\gamma}' > \hat{\gamma}$. For simplicity of notation, in the remaining part, we use h rather than $c_{L'}$, i.e., $c_{L'} = h$. The most straightforward way to approach this issue is to solve

$$(\mathrm{NP}): \begin{cases} \max\limits_{\gamma', \alpha', r, \hat{r}, u_i, \hat{u}_i, w_i, \hat{w}_i \, i \in \mathcal{I}', h} & \gamma' \\ \text{subject to} & (18), (19), \ h \in \mathbf{R}^n, \end{cases}$$

where E'_i and P'_i are nonlinear and discontinuous functions of h.

We approach this problem considering (NP) as a set of subproblems. Each subproblem corresponds to the set of polytopes which are divided by a hyperplane $\mathcal{H}(h)$. Suppose that $\mathcal{H}(h)$ is given and it does not intersect any $x_{i,j} \in (\mathcal{X}_i \backslash \{0\})$. Then, we can determine

$$\hat{s}_{i,j,L'} = \begin{cases} 1, & \text{if } h^\top x_{i,j} > 0, \\ -1, & \text{if } h^\top x_{i,j} < 0, \end{cases} \tag{20}$$

for every $x_{i,j} \in (\mathcal{X}_i \backslash \{0\})$, where $i \in \mathcal{I}$. On the contrary, suppose that $\mathcal{H}(h)$ is not given but $\{\hat{s}_{i,j,L'}, \ j = 1, 2, \cdots, |(\text{ node } \mathcal{X}_i \backslash \{0\})|, \ i \in \mathcal{I}\}$ is given. We say that $\{\hat{s}_{i,j,L'}\}$ is feasible if

$$\mathrm{LP}(\{\hat{s}_{i,j,L'}\}) : \begin{cases} \max\limits_{h} & 1 \\ \text{subject to} & (21) \end{cases}$$

has a feasible solution, where

$$\hat{s}_{i,j,L'} x_{i,j}^\top h > 0, \quad x_{i,j} \in (\text{ node } \mathcal{X}_i \backslash \{0\}), \quad i \in \mathcal{I}. \tag{21}$$

Proposition 1 *Suppose that a feasible* $\{\hat{s}_{i,j,L'}\}$ *is given and let* h *be an arbitrary feasible solution of* $\mathrm{LP}(\{\hat{s}_{i,j,L'}\})$. *If* $\hat{s}_{i,j,L'} \hat{s}_{i,j',L'} = 1$ *for all* $x_{i,j}, \ x_{i,j'} \in (\text{ node } \mathcal{X}_i \backslash \{0\})$, *then* $\mathcal{X}_i$ *is not divided by the hyperplane* $\mathcal{H}(h)$, *and* $s_{i,L'} = \hat{s}_{i,j,L'}$. *Otherwise* $\mathcal{X}_i$ *is divided into* $\mathcal{X}_{i_1}$ *and* $\mathcal{X}_{i_2}$, *by* $\mathcal{H}(h)$ *and* $s_{i_1,L'} = 1$ *and* $s_{i_2,L'} = -1$. *Moreover, for any* $\mathcal{X}'_i = \mathcal{X}_{p(i)}$ *which is not divided by* $\mathcal{H}(h)$, *we have* $s_{i,\ell} = s_{p(i),\ell}$ *for all* $\ell \in calL$ *and* $s_{i,L'} = \hat{s}_{p(i),j,\text{Ł}'}$ *for any* $x_{i,j} \in \mathcal{X}'_{p(i)}$

Suppose that a feasible $\{\hat{s}_{i,j,L'}\}$ is given, and, hence, $\{s_{i,L'}\}$ are determined for all $\mathcal{X}_i$ by Proposition 1. For simplicity of notation, as we supposed, $\mathcal{X}_i = \mathcal{X}'_i$ for $i = 1, 2, \cdots, I-J$, and $\mathcal{X}_i = \mathcal{X}'_i \cup \mathcal{X}'_{i+J}$ for $i = I-J+1, I-J+2, \cdots, I$. Then, the corresponding subproblem is given by

$$\mathrm{NP}(\{\hat{s}_{i,j,L'}\}): \begin{cases} \max\limits_{\gamma', \alpha', r, \hat{r}, u_i, \hat{u}_i, w_i, \hat{w}_i, i \in \mathcal{I}', h} \quad \gamma' \\ \text{subject to} \\ \quad (18), (19), (21), \ h \in \mathbf{R}^n, \ \ |h| = 1, \\ \quad R = \hat{R}, \ \ U_i = \hat{U}_i, \ \ W_i = \hat{W}_i, \\ \quad i \in \mathcal{I}, \ \ q \in \mathcal{Q} \end{cases}$$

In (18), E'_i's are matrix linear functions of h since $\{s_{i,L'}\}$ is determined. But, $P'_i = E_i'^\top R' E'_i$ is given by

$$P'_i = \hat{P}_{p(i)} + s_{i,L'}[E_{p(i)}^\top r h^\top + (E_{p(i)}^\top r h^\top)^\top] + \hat{r} h h^\top, \tag{22}$$

and, hence, P'_i has quadratic term $hh^\top$ and bilinear terms $rh^\top$. These quadratic or bilinear terms are included in constraints of $\mathrm{NP}(\{\hat{s}_{i,j,L'}\})$, and $\mathrm{NP}(\{\hat{s}_{i,j,L'}\})$ is also difficult to solve.

We change our strategy. We have the following.

Proposition 2 *Let* $(\hat{\gamma}, \hat{\alpha}, \hat{R}, \{\hat{U}_i\}, \{\hat{W}_i\})$ *be an optimal solution of* (LMI). *For* $i \in \mathcal{I}$ *and* $q \in \mathcal{Q}$, *let us consider*

$$(\mathrm{QP0})_i : \begin{cases} \max\limits_{x} \ x^\top \hat{Q}_i x \\ \text{subject to} \quad x \in \mathcal{X}_i, \end{cases}$$

where

$$\hat{Q}_i = A_{mq}^\top \hat{P}_i + \hat{P}_i A_{mq} + E_i^\top \hat{U}_i E_i + \hat{\gamma} I, \tag{23}$$

and m is the suffix such that $\mathcal{X}_i \subseteq \mathcal{X}_m$.

Let $\hat{x}_i$ be an optimal solution of $(\mathrm{QP0})_i$*, and let*

$$\hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}} = \max_{i\in\mathcal{I},\, q\in Q} \hat{x}_i^\top \hat{Q}_i \hat{x}_i, \quad (24)$$

If $\hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}} < 0$, then $V(x)$ is a Lyapunov function for (1).

Suppose that $\hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}} \geq 0$ and that a feasible $\{\hat{s}_{i,j,L'}\}$ is given. We seek a vector h such that $|h| = 1$ and $\hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}'_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}} < \hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}}$ for some $u_{\hat{i}\hat{q}} \geq 0$, $\hat{u}_{\hat{i}\hat{q}} \geq 0$, $r \in \mathbf{R}^L$ and $\hat{r} \in R$, where

$$\hat{x}_{\hat{i}\hat{q}}^\top Q'_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}} = \hat{x}_{\hat{i}\hat{q}}^\top A_{mq}^\top P'_{\hat{i}} \hat{x}_{\hat{i}\hat{q}} + \hat{x}_{\hat{i}\hat{q}}^\top E'^\top_{\hat{i}} U'_{\hat{i}\hat{q}} E'_{\hat{i}} \hat{x}_{\hat{i}\hat{q}} + \hat{\gamma}|\hat{x}_{\hat{i}\hat{q}}|^2, \quad (25)$$

in which $P'_{\hat{i}}$ is given by (22), $U_{\hat{i}\hat{q}} = \hat{U}_{\hat{i}\hat{q}}$, and $R = \hat{R}$.

Once we have such a h, we divide $\mathcal{X}_i$'s by $\mathcal{H}(h)$ and solve (LMI)' to get a better PQLF candidate.

Theorem 2 *There exist h such that $\hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}'_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}} < \hat{x}_{\hat{i}\hat{q}}^\top \hat{Q}_{\hat{i}\hat{q}} \hat{x}_{\hat{i}\hat{q}}$ for some $u_{\hat{i}\hat{q}} \geq 0$, $\hat{u}_{\hat{i}\hat{q}} \geq 0$, $r \in \mathbf{R}^L$ and $\hat{r} \in R$ if and only if there exist h such that $g_{\hat{i}\hat{q}}(h, r, \hat{r}) < 0$ for some $r \in \mathbf{R}^L$ and $\hat{r} \in R$, where*

$$g_{\hat{i}\hat{q}}(h, r, \hat{r}) = s_{\hat{i},L'}[\hat{y}_{\hat{i}\hat{q}}^\top E_{p(i)}^\top r(h^\top \hat{x}_{\hat{i}\hat{q}}) + (\hat{y}_{\hat{i}\hat{q}}^\top h) r^\top E_{p(i)} \hat{x}_{\hat{i}\hat{q}}] + \hat{r}(\hat{y}_{\hat{i}\hat{q}}^\top h)(h^\top \hat{x}_{\hat{i}\hat{q}}), \quad (26)$$

and $\hat{y}_{\hat{i}\hat{q}} = A_{mq} \hat{x}_{\hat{i}\hat{q}}$.

Note that $g_{\hat{i}\hat{q}}(h, r, \hat{r})$ includes a quadratic term $(\hat{y}_{\hat{i}\hat{q}}^\top h)(h^\top \hat{x}_{\hat{i}\hat{q}})$ and bilinear terms $r(h^\top \hat{x}_{\hat{i}\hat{q}})$ and $(\hat{y}_{\hat{i}\hat{q}}^\top h) r^\top$. However, $(h^\top \hat{x}_{\hat{i}\hat{q}})$ and $(\hat{y}_{\hat{i}\hat{q}}^\top h)$ are scalars, and, it is rather easy to treat them than the case when quadratic or bilinear terms are included in P'_i. Define a polytope $\mathcal{P}(\{\hat{s}_{i,j,L'}\})$ by

$$\mathcal{P}(\{\hat{s}_{i,j,L'}\}) = \{h \in \mathbf{R}^L : |h_i| \leq 1, \ i = 1, 2, \cdots, L, \ \hat{s}_{i,j,L'} x_{i,j}^\top h \geq 0, \ x_{i,j} \in \text{node } \mathcal{X}_i, \ i \in \mathcal{I}\},$$

where h_i denotes the i-th element of h. Moreover, we define

$$\xi_{\max} = \max\{\hat{x}_{\hat{i}\hat{q}}^\top h, \ h \in \text{node } \mathcal{P}(\{\hat{s}_{i,j,L'}\}), \ h \neq 0\}, \quad (27)$$

$$\xi_{\min} = \min\{\hat{x}_{\hat{i}\hat{q}}^\top h, \ h \in \text{node } \mathcal{P}(\{\hat{s}_{i,j,L'}\}), \ h \neq 0\}, \quad (28)$$

$$\eta_{\max} = \max\{\hat{y}_{\hat{i}\hat{q}}^\top h, \ h \in \text{node } \mathcal{P}(\{\hat{s}_{i,j,L'}\}), \ h \neq 0\}, \quad (29)$$

$$\eta_{\min} = \min\{\hat{y}_{\hat{i}\hat{q}}^\top h, \ h \in \text{node } \mathcal{P}(\{\hat{s}_{i,j,L'}\}), \ h \neq 0\}. \quad (30)$$

Let $\xi = (\hat{x}_{\hat{i}\hat{q}}^\top h)$ and let $\eta = (\hat{y}_{\hat{i}\hat{q}}^\top h)$. For each (ξ, η) satisfying

$$\xi_{\min} \leq \xi \leq \xi_{\max} \quad \eta_{\min} \leq \eta \leq \eta_{\max} \quad (31)$$

we solve

$$(\mathrm{CP2})(\xi,\eta) : \begin{cases} \min_{\sigma,h} & \sigma \\ \text{subject to} & \xi - \delta \leq \hat{x}_{\hat{i}\hat{q}}^\top h \leq \xi + \delta, \\ & \eta - \delta \leq \hat{y}_{\hat{i}\hat{q}}^\top h \leq \eta + \delta, \\ & \begin{bmatrix} 1 & h^\top \\ h & I \end{bmatrix} \leq 0, \quad \begin{bmatrix} \sigma & h^\top \\ h & I \end{bmatrix} \geq 0. \end{cases}$$

where $\delta > 0$ is a small positive number.

Let (σ^*, h^*) be the optimal solution of $(\mathrm{CP2})(\xi, \eta)$. If $\sigma^* = 1$, then we have a h^* such that $|\hat{x}_{\hat{i}\hat{q}}^\top h^* - \xi| \leq \delta$, $|\hat{y}_{\hat{i}\hat{q}}^\top h^* - \eta| \leq \delta$, and $|h^*| = 1$, and we solve

$$(\mathrm{LP2})(\xi,\eta) : \begin{cases} \min_{r,\hat{r}} & \tilde{g}(r, \hat{r}; \xi, \eta) \\ \text{subject to} & |r_i| \leq 1, \quad i = 1, 2, \cdots, L, \\ & |\hat{r}| \leq 1, \end{cases}$$

where $\tilde{g}(r, \hat{r}; \xi, \eta) = s_{\hat{i},L'}(\xi \hat{y}_{\hat{i}\hat{q}}^\top + \eta \hat{x}_{\hat{i}\hat{q}}^\top) E_{p(i)}^\top \ r + \eta\xi \ \hat{r}$ and r_i denotes the i-th component of $r \in \mathbf{R}^L$.

Let $(r^*, \hat{r}^*)$ be the optimal solution of $(\mathrm{LP2})(\xi, \eta)$. If $\tilde{g}(r^*, \hat{r}^*; \xi, \eta) < 0$, then we use h^* as $c_{L'}$. If $\tilde{g}(r^*, \hat{r}^*; \xi, \eta) \geq 0$, then we repeat above process for other (ξ, η). If $\tilde{g}(r^*, \hat{r}^*; \xi, \eta) \geq 0$ for all (ξ, η) satisfying (31), we fail to construct PQLF for the system (1). It might be better to repeat above process for other (ξ, η) even if $\tilde{g}(r^*, \hat{r}^*; \xi, \eta) < 0$.

4. Conclusion

In this paper, we proposed a method to increase the freedom included in PQLFs for getting less conservative stability results for uncertain systems. Because of the limitation of spaces, we can not include examples to demonstrate the usefulness of the proposed method. But readers who are interested this results can be obtained more details including examples from authors upon request. Theorem 1 is suggested by Dr. Masubuchi. We appliciate it to him.

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