2008 International Symposium on Nonlinear Theory and its Applications
NOLTA'08, Budapest, Hungary, September 7-10, 2008

Geometric aspects of a certain type of nonlinear diffusion equations

Atsumi Ohara† and Tatsuaki Wada‡

†Department of Systems Science, Osaka University, 1-3 Machikaneyama, Toyonaka, Osaka, 560-8531, Japan
‡Department of Electrical and Electronic Engineering, Ibaraki University, Hitachi, Ibaraki 316-8511, Japan
Email: ohara@sys.es.osaka-u.ac.jp, wada@mx.ibaraki.ac.jp

Abstract—This paper presents new geometric aspects of the behaviors of solutions to the porous medium equation (PME) and its associated equation. First we discuss the Legendre structure with information geometry on the manifold of generalized exponential densities. Next by equipping the so-called q-Gaussian densities with such structure, we show several physically and geometrically interesting properties of the solutions, e.g., characterization of the moment-conserving projection of a solution, evaluations of evolutional velocities of the second moments and the convergence rate to the manifold in terms of the geodesic curves, divergence and so on.

1. Introduction

Let $u(x,t)$ and $p(x,\tau)$ on $\mathbf{R}^n \times \mathbf{R}_+$ be, respectively, the solutions of the following nonlinear diffusion equation, which is called the *porous medium equation* (PME):

$$\frac{\partial u}{\partial t} = \Delta u^m, \quad m > 1 \tag{1}$$

with nonnegative initial data $0 \le u(x,0) = u_0(x) \in L^1(\mathbf{R}^n)$, and the associated nonlinear Fokker-Planck equation (NFPE):

$$\frac{\partial p}{\partial \tau} = \nabla \cdot (\beta x p + D \nabla p^m), \quad \beta > 0 \tag{2}$$

with nonnegative initial data $0 \le p(x,0) = p_0(x) \in L^1(\mathbf{R}^n)$. Here, D is a real symmetric positive definite matrix, which represents the diffusion coefficients. As is widely known [16, 17] and shown later, one solution is obtained from a simple transformation from the other, and vice versa.

The PME and NFPE with $m > 1$ represent the so-called *slow diffusion* phenomena, which naturally arises in many physical problems including percolation of a fluid through porous media and so on. See for [1, 2, 3, 4, 5] and the references therein. Hence the behaviors of their solutions have been extensively studied in both analytical and thermostatistical aspects in the literature [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16], just to name a few.

In Section 2 we introduce the Legendre structure on the space of generalized exponential density functions, following [23, 24, 21, 22] which is compatible to information geometry [19, 20] on the space. The main results on behaviors of solutions in terms of induced geometric concepts are described in Section 3. The manifold of q-Gaussian densities, which is invariant for the equation, plays an central role. Evolutions of the second moments, the convergence rate to the manifold are discussed.

2. Generalized exponential family and its Legendre structure

For a fixed strictly increasing and positive function $\phi(s)$ on $(0,\infty)$, define the *generalized logarithmic function* as follows:

$$\ln_\phi(t) := \int_1^t \frac{1}{\phi(s)} ds, \quad t > 0.$$

The *generalized exponential function* denoted by $\exp_\phi$ is defined as the inverse function of $\ln_\phi$.

Define a convex function $F_\phi(s)$ for $s > 0$ by

$$F_\phi(s) := \int_1^s \ln_\phi t dt, \quad F_\phi(0) < +\infty \text{ :assumed.} \tag{3}$$

For probability density functions $p(x)$ and $q(x)$, introduce a *generalized entropy* functional defined by

$$\mathcal{I}_\phi[p] := \int -F_\phi(p(x)) + (1 - p(x))F_\phi(0) dx, \tag{4}$$

and the *Bregman divergence* defined by

$$\mathcal{D}_\phi[p\|q] := \int U_\phi(\ln_\phi q) - U_\phi(\ln_\phi p) - p(\ln_\phi q - \ln_\phi p) dx, \tag{5}$$

where the function U_ϕ is defined by

$$U_\phi(t) := t \exp_\phi t - F_\phi(\exp_\phi t). \tag{6}$$

Let us consider the following finite dimensional statistical model called *the generalized exponential family* [23] or *U-statistical model* [21], which is defined by

$$\mathcal{M}_\phi = \{p_\theta(x) = \exp_\phi(\theta^T h(x) - \kappa_\phi(\theta)) | \theta \in \Omega \subset \mathbf{R}^d\} \subset L^1(\mathbf{R}^n),$$

where $h(x) = (h_i(x)), i = 1, \cdots, d$ is a certain vector-valued function and $\kappa_\phi(\theta)$ is a normalizing factor of $p_\theta(x)$.

Introduce the following *potential function*:

$$\Psi_\phi(\theta) := \int U_\phi(\ln_\phi p_\theta) + (1 - p_\theta)F_\phi(0) dx + \kappa_\phi(\theta).$$

It follows from the relation $\exp_\phi = U'_\phi$ that

$$\eta_i(\theta) := \partial_i \Psi_\phi(\theta) = \int h_i(x) p_\theta(x) d\mu = \mathbf{E}_{p_\theta}[h_i(x)], \tag{7}$$

where $\partial_i := \partial/\partial\theta^i$ and we denote by $\mathbf{E}_p[\cdot]$ the expectation operator for the density p. Then, the Hesse matrix of $\Psi_\phi(\theta)$ is expressed by

$$\partial_i \partial_j \Psi_\phi(\theta) = \int \tilde{h}_i(x) \exp'_\phi(\theta^T h(x) - \kappa_\phi(\theta)) \tilde{h}_j(x) dx, \tag{8}$$

where $\tilde{h}_i(x) := h_i(x) - \partial_i \kappa_\phi(\theta)$. We see that it is positive semidefinite because $\exp'_\phi$ is positive, and hence, Ψ_ϕ

is a convex function of θ. In the sequel, we assume that $(\partial_i\partial_j\Psi_\phi) = (\partial\eta_j/\partial\theta^i)$ is positive definite for $\forall\theta \in \Omega$. Hence, $\eta = (\eta_j)$ is bijective to (θ^i) and we call $\theta = (\eta_i)$ the *expectation coordinate system* for $\mathcal{M}_\phi$. By the relation (7) the Legendre conjugate of $\Psi_\phi(\theta)$ is the sign-reversed generalized entropy of $p_\theta \in \mathcal{M}_\phi$, i.e,

$$\Psi^*_\phi(\eta) = \theta^T\eta - \Psi_\phi(\theta) = -\mathcal{I}_\phi[p_\theta]. \quad (9)$$

Hence, $\Psi_\phi(\theta)$ can be physically interpreted as the *generalized Massieu potential* [26, 25] and our Riemmanian metric $(\partial_i\partial_j\Psi_\phi) = (\partial\eta_j/\partial\theta^i)$ intoduced below is regarded as a *susceptance* matrix.

As a *Riemannian metric* $g = (g_{ij})$ on $\mathcal{M}_\phi$, which is an inner product for tangent vectors, we use the Hesse matrix of Ψ_ϕ. Note that we can alternatively express (8) as

$$g_{ij}(\theta) = g(\partial_i, \partial_j) := \partial_i\partial_j\Psi_\phi = \int \partial_i p_\theta \partial_j \ln_\phi p_\theta dx.$$

Further we define the *mixture connection* $\nabla^{(\mathrm{m})}$ and *generalized exponential connection* $\nabla^{(\mathrm{ge})}$ by their components

$$\Gamma^{(\mathrm{m})}_{ij,k}(\theta) = g(\nabla^{(\mathrm{m})}_{\partial_i}\partial_j, \partial_k) := \int \partial_i\partial_j p_\theta \partial_k \ln_\phi p_\theta dx,$$

$$\Gamma^{(\mathrm{ge})}_{ij,k}(\theta) = g(\nabla^{(\mathrm{ge})}_{\partial_i}\partial_j, \partial_k) := \int \partial_k p_\theta \partial_i\partial_j \ln_\phi p_\theta dx. \quad (10)$$

Then the duality relation of the connections [19, 20] holds, i. e., $\partial_i g_{jk} = \Gamma^{(\mathrm{m})}_{ij,k} + \Gamma^{(\mathrm{ge})}_{ik,j}$. Further, $\mathcal{M}_\phi$ can be proved to be flat with respect to both $\nabla^{(\mathrm{m})}$ and $\nabla^{(\mathrm{ge})}$. Thus, we have obtained *dually flat* [20] structure $(g, \nabla^{(\mathrm{m})}, \nabla^{(\mathrm{ge})})$ on $\mathcal{M}_\phi$ defined by the derivatives of Ψ_ϕ.

Proposition 1 *Let C be a one-dimensional submanifold on $\mathcal{M}_\phi$. If C is expressed as a straight line in the coordinates θ, then C coincides with a $\nabla^{(\mathrm{m})}$-geodesic (*m-geodesic*, in short) curve. If C is expressed as a straight line in the coordinates η, then C coincides with a $\nabla^{(\mathrm{ge})}$-geodesic (*ge-geodesic*) curve.*

Definition 1 *Let $p(x)$ be a given density. If there exists the minimizing density function $\hat{p}_\theta(x)$ for the variational problem* $\min_{p_\theta\in\mathcal{M}_\phi} \mathcal{D}_\phi[p\|p_\theta]$*, or equivalently, the minimizing parameter $\hat{\theta}$ for the problem* $\min_{\theta\in\Omega} \mathcal{D}_\phi[p\|p_\theta]$ *exists, we call $\hat{p}_\theta(x) = p_{\hat{\theta}}(x)$ the* m-projection of $p(x)$ to $\mathcal{M}_\phi$.

Proposition 2 *Let $\hat{p}_\theta \in \mathcal{M}_\phi$ be the m-projection of p. Then the following properties hold:*

i) *The expectation of $h(x)$ is conserved by the m-projection, i.e., $\mathbf{E}_p[h(x)] = \mathbf{E}_{\hat{p}_\theta}[h(x)]$,*

ii) *The following triangular equality holds: $\mathcal{D}_\phi[p\|p_\theta] = \mathcal{D}_\phi[p\|\hat{p}_\theta] + \mathcal{D}_\phi[\hat{p}_\theta\|p_\theta]$ for all $p_\theta \in \mathcal{M}_\phi$.*

Remark 1 *From the statement i) the m-projection $\hat{p}_\theta$ is characterized as the density in $\mathcal{M}_\phi$ with the equal expectation of $h(x)$ to that for p. Note that the following relation:*

$$\begin{aligned}\mathcal{D}_\phi[p\|\hat{p}_\theta] &= \Psi_\phi(\hat{\theta}) - \mathcal{I}_\phi[p] - \hat{\theta}^T\mathbf{E}_p[h(x)] \\ &= \Psi_\phi(\hat{\theta}) - \hat{\theta}^T\hat{\eta} - \mathcal{I}_\phi[p] = \mathcal{I}_\phi[\hat{p}_\theta] - \mathcal{I}_\phi[p] \geq 0.\end{aligned}$$

Thus, $\hat{p}_\theta$ achieves the maximum entropy among densities with the equal expectation of $h(x)$.

3. Several geometric properties of the porous medium and the associated Fokker-Planck equation

Set $\phi(u) = u^q, q > 0, q \neq 1$, then we have the q-logarithmic and exponential functions [18]:

$$\begin{aligned}\ln_\phi t = \ln_q t &:= (t^{1-q} - 1)/(1-q), \\ \exp_\phi t = \exp_q t &:= [1 + (1-q)t]_+^{1/(1-q)}.\end{aligned}$$

Consider the q-Gaussian density function defined by:

$$f(x;\theta,\Theta) = \exp_q\left(\theta^T x + x^T\Theta x - \kappa(\theta,\Theta)\right), \quad (11)$$
$$\theta = (\theta^i) \in \mathbf{R}^n, \Theta = (\theta^{ij}) \in \mathbf{R}^{n\times n},$$

where Θ is a real symmetric negative definite matrix and $\kappa(\theta,\Theta)$ is a normalizing constant. We denote by $\mathcal{M}$ the set of q-Gaussian densities, i.e.,

$$\mathcal{M} := \left\{ f(x;\theta,\Theta) | \, \theta \in \mathbf{R}^n, \; 0 > \Theta = \Theta^T \in \mathbf{R}^{n\times n} \right\}.$$

For this setting, the corresponding generalized entropy and divergence are

$$\mathcal{I}[p] = \frac{1}{2-q}\int \frac{p(x)^{2-q} - p(x)}{q-1} dx \quad (12)$$

$$\mathcal{D}[p\|q] = \int \frac{q(x)^q - p(x)^q}{q} - p(x)\frac{q(x)^{q-1} - p(x)^{q-1}}{q-1} dx, \quad (13)$$

In the sequel we fix the relation between the exponents of the PME and the parameter of q-exponential function by $m = 2 - q$. Hence, we consider the case $1 < m < 2$, or equivalently, $0 < q < 1$. Since we fix $\phi(u) = u^q$, we omit the subscripts ϕ used to denote several quantities. By a suitable linear scaling of t we can consider the problem by fixing β to an arbitrary constant. Hence, we fix β and introduce another constant α for notational simplicity as follows:

$$\beta = \frac{1}{n(m-1)+2}, \quad \alpha = n\beta.$$

For the q-Gaussian family $\mathcal{M}$, we can regard (θ,Θ) as the canonical coordinates, and the first moment vector and second moment matrix (η, H) defined by

$$\eta = \int xp(x;\theta,\Theta)dx, \quad H = \int xx^T p(x;\theta,\Theta)dx,$$

as the expectation coordinates, respectively.

We assume the $u(x,0)$ and $p(x,0)$, which denote initial data of the PME and the NFPE, are nonnegative and integrable function with finite second moments. When we consider the set of solutions, we restrict their initial masses to be normalized to one without loss of generalities.

It is proved that there exists a unique nonnegative week solution if $m > 0$ [16, Theorem 5.1], and that the mass $\int u(x,t)dx$ is invariant for all $t > 0$ if $m \geq (n-2)/n$ [16].

First of all, we review how the solutions of PME and NFPE relate in the proposition below. Because of this fact the properties of the solution of PME (1) are important to investigate those of NFPE (2) and vise versa.

Proposition 3 *Let $u(x,t)$ be a solution of the PME (1) with initial data $u(x,0) = u_0(x) \in L^1(\mathbf{R}^n)$, Define*

$$p(z,\tau) := (t+1)^\alpha u(x,t), \quad z := (t+1)^{-\beta}Rx, \;\; \tau := \ln(t+1),$$

then $p(z,\tau)$ is a solution of (2) with $\nabla = \nabla_z$, $D = RR^T$ and initial data $p(z,0) = u_0(R^{-1}z)$.

Next, we find that the equilibrium density for the NFPE is on the q-Gaussian family $\mathcal{M}$ via Lyapunov approach. To analyze the behavior of (2) let us define *generalized free energy*:

$$\mathcal{F}[p] := \int \frac{\beta}{2m} x^T D^{-1} x p(x) dx - \mathcal{I}[p]$$

This type of functional was first introduced in [8, 9]. We have

$$\frac{d\mathcal{F}[p(x,\tau)]}{d\tau} = -\int p|\beta R^{-1}x + (2-q)p^{-q} R\nabla p|^2 dx \le 0. \tag{14}$$

Thus, the equilibrium density $p_\infty(x)$ is determined from (14) as a q-Gaussian:

$$p_\infty(x) = f(x; 0, \Theta_\infty) = \exp_q(x^T \Theta_\infty x - \kappa(0, \Theta_\infty)), \tag{15}$$

where the canonical parameters are given by

$$\theta_\infty = 0, \quad \Theta_\infty = -\frac{\beta}{2m} D^{-1}.$$

Note that we can express the difference of the free energies at $p(x)$ and the equilibrium $p_\infty(x) \in \mathcal{M}$ by the divergence:

$$\begin{aligned} \mathcal{D}[p \| p_\infty] &= \Psi(0, \Theta_\infty) - \mathcal{I}[p] - \Theta_\infty \cdot \mathbf{E}_p[xx^T] \\ &= \mathcal{F}[p] - \mathcal{F}[p_\infty]. \end{aligned}$$

Thus, the minimization of $\mathcal{F}[\cdot]$ is equivalent to that of $\mathcal{D}[\cdot \| p_\infty]$.

Finally, we show the q-Gaussian family is an invariant manifold for PME and NFPE. Since it follows from direct calculations, we omit the proof.

Proposition 4 *The q-Gaussian family $\mathcal{M}$ is an invariant manifold for both PME and NFPE.*

3.1. Trajectories of m-projections

Let $\eta^{\mathrm{PM}} = (\eta_i^{\mathrm{PM}})$ and $H^{\mathrm{PM}} = (\eta_{ij}^{\mathrm{PM}})$ be, respectively, the first moment vector and the second moment matrix, i.e.,

$$\eta_i^{\mathrm{PM}}(t) := \mathbf{E}_u[x_i] = \int x_i u(x,t) dx, \quad \eta_{ij}^{\mathrm{PM}}(t) := \mathbf{E}_u[x_i x_j].$$

Theorem 1 *Consider solutions of the PME with the common initial first and second moments. Then their m-projections to $\mathcal{M}$ evolve monotonically along with the common m-geodesic curve that starts the density decided by the initial moments.*

Outline of the proof) Differentiating η_{ij}^{PM} by t, we see that the second moments evolves as

$$\begin{aligned} \eta_{ij}^{\mathrm{PM}}(t) &= \eta_{ij}^{\mathrm{PM}}(0) + \delta_{ij}\sigma_u^{\mathrm{PM}}(t), \\ \sigma_u^{\mathrm{PM}}(t) &:= 2\int_0^t dt' \int u(x,t')^m dx. \end{aligned}$$

Note that $\sigma_u^{\mathrm{PM}}(t)$ is positive and monotone increasing on $t > 0$. By similar argument we see that $\dot{\eta}_i^{\mathrm{PM}} = 0$, i.e., the first moment vector is invariant. From the fact that the m-projection conserves moments, Proposition 2 and Proposition 1 the statement follows. Q.E.D.

Remark 2 *i) From the argument for NFPE, we will see that $\sigma_u^{\mathrm{PM}}(t) = O(t^{2\beta})$ as $t \to \infty$.*
ii) Theorem implies that the trajectories of m-projections on $\mathcal{M}$ for all the PME solutions $u(x,t)$ are parallelized in the expectation coordinates, i.e.,

$$\eta^{\mathrm{PM}}(t) = \eta^{\mathrm{PM}}(0), \tag{16}$$
$$H^{\mathrm{PM}}(t) = H^{\mathrm{PM}}(0) + \sigma_u^{\mathrm{PM}}(t) I. \tag{17}$$

Thus, the PME has the following constants of motion*:*

$$\begin{aligned} I_0 &= \int u(x,t)dx, \quad I_i = \int x_i u(x,t) dx, \quad i = 1, \cdots, n, \\ I_{ij} &= \int x_i x_j u(x,t) dx, \quad i = 1, \cdots, n, \ j = 1, \cdots, n, \ i \neq j, \\ I_{kk} &= \sum_{i=1}^n e_i^{(k)} \left(\int x_i^2 u(x,t) dx - \eta_{ii}(0) \right), \quad k = 1, \cdots, n-1, \end{aligned}$$

where $e^{(k)} = (e_1^{(k)} \cdots e_n^{(k)}), k = 1, \cdots, n-1$ are a set of $n-1$ basis vectors of the hyperplane $\mathcal{H} = \{x \in \mathbf{R}^n | \sum_{i=1}^n x_i = 0\}$.

Let $\hat{f}_0(x) \in \mathcal{M}$ be the m-projection of the density $f_0(x)$. Consider two solutions $u_1(x,t)$ and $u_2(x,t)$ of PME satisfying $u_1(x,t_0) = f_0(x)$ and $u_2(x,t_0) = \hat{f}_0(x)$ for some $t = t_0$. From the moment conservation property of the m-projection stated in Proposition 2, the second moment matrices $H_i^{\mathrm{PM}}(t)$ of $u_i(x,t)$ for $i = 1, 2$ satisfy $H_1^{\mathrm{PM}}(t_0) = H_2^{\mathrm{PM}}(t_0)$. However, their velocities at t_0 have the relation:

$$\begin{aligned} \dot{H}_1^{\mathrm{PM}}(t_0) - \dot{H}_2^{\mathrm{PM}}(t_0) &= 2\int f_0^m(x) - \hat{f}_0^m(x) dx \, I \\ &= 2m(m-1)\left(\mathcal{I}[\hat{f}_0] - \mathcal{I}[f_0]\right) I \end{aligned}$$

from (17) and the expression of the generalized entropy (12). Using the relation in Remark 1, we have the following:

Corollary 1 *Let $\hat{f}_0(x) \in \mathcal{M}$ be the m-projection of a density $f_0(x)$ and assume that two solutions $u_1(x,t)$ and $u_2(x,t)$ of PME satisfy the conditions $u_1(x,t_0) = f_0(x)$ and $u_2(x,t_0) = \hat{f}_0(x)$ at some $t = t_0$. Then velocities of their respective second moment matrices at t_0 are related by*

$$\dot{H}_1^{\mathrm{PM}}(t_0) - \dot{H}_2^{\mathrm{PM}}(t_0) = 2m(m-1)\mathcal{D}[f_0 \| \hat{f}_0] I.$$

Thus, the m-projection $\hat{u}_1(x,t)$ of $u_1(x,t) \notin \mathcal{M}$, which has the common second moment matrix $H_1^{\mathrm{PM}}(t)$ for all t, evolves faster than $u_2(x,t) \in \mathcal{M}$, while $\hat{u}_1(x,t)$ and $u_2(x,t)$ have the common trajectory on $\mathcal{M}$ by Theorem 1. The corollary suggests that by measuring the diagonal elements of $H_1^{\mathrm{PM}}(t)$ we can estimate how far $u_1(x,t)$ is from $\mathcal{M}$ in terms of the divergence. Note that the difference of velocities vanishes when $m \to 1$. Hence, this is the specific property of the slow diffusions governed by the PME.

Let $\eta^{\mathrm{FP}}(\tau)$ and $H^{\mathrm{FP}}(\tau)$ be, respectively, the first and the second moments of $p(x,\tau)$, i.e.,

$$\eta^{\mathrm{FP}} = \mathbf{E}_p[x], \quad H^{\mathrm{FP}} = \mathbf{E}_p[xx^T].$$

From the behavior of the moments of the PME and the above relations of moments, we have

$$\begin{aligned} \eta^{\mathrm{FP}}(\tau) &= e^{-\beta\tau}\eta^{\mathrm{FP}}(0), \\ H^{\mathrm{FP}}(\tau) &= e^{-2\beta\tau} H^{\mathrm{FP}}(0) + e^{-2\beta\tau}\sigma_p^{\mathrm{FP}}(e^\tau - 1) D, \end{aligned}$$

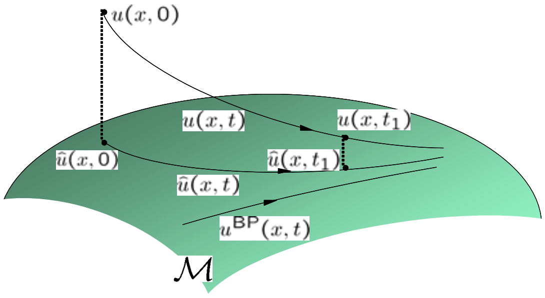

Figure 1: A solution $u(x,t)$ of the PME, its m-projection $\hat{u}(x,t)$ and the Barenblatt-Pattle solution $u^{\mathrm{BP}}(x,t)$ on $\mathcal{M}$

where the scaling $\tau = \ln(t+1)$ is assumed and $\sigma_p^{\mathrm{FP}}(t)$ is defined by

$$\begin{aligned}\sigma_p^{\mathrm{FP}}(t) &:= 2\int_0^{\ln(1+t)} d\tau' e^{\tau'+\alpha(1-m)\tau'} \int p(x,\tau')^m dx \\ &= \det(R)\sigma_u^{\mathrm{PM}}(t).\end{aligned}$$

for a solution u of the PME and the corresponding solution p of the NFPE. Note that differentiating the above by t, we have the relation:

$$(1+t)^{\alpha(1-m)} \int p(z,\tau)^m dz = \det(R) \int u(x,t)^m dx. \quad (18)$$

For the limiting case $m \to 1$ (and accordingly $\beta \to 1/2$), we see that the above expressions recover the well-known linear Fokker-Plank case with a drift vector $x/2$:

$$\eta^{\mathrm{FP}}(\tau) = e^{-\tau/2}\eta^{\mathrm{FP}}(0), \quad H^{\mathrm{FP}}(\tau) = e^{-\tau}H^{\mathrm{FP}}(0) + 2(1-e^{-\tau})D.$$

Since we know that $p(x,\tau)$ converges to $p_\infty(x) \in \mathcal{M}$ in (15) and it holds that

$$\lim_{\tau\to\infty} H^{\mathrm{FP}}(\tau) = \sqrt{\det D}\left(\lim_{t\to\infty}(t+1)^{-2\beta}\sigma_u^{\mathrm{PM}}(t)\right)D \quad (19)$$

because $\det R = \sqrt{\det D}$, we conclude that the left-hand side of (19) exists and $\sigma_u^{\mathrm{PM}}(t) = O(t^{2\beta})$ as $t \to \infty$ (Cf. Remark 2). Summing up the above with Proposition 1, we obtain the following geometric property of the NFPE:

Corollary 2 *Consider solutions of the NFPE with the common initial first and second moments. Then their m-projections to $\mathcal{M}$ evolve along with the common m-geodesic curve connecting the density of the initial m-projection and the equilibrium $p_\infty(x)$.*

Note that the following relation holds with the scaling $\tau = \ln(t+1)$:

$$\frac{d}{d\tau}H^{\mathrm{FP}}(\tau) = (t+1)^{-2\beta}\left(-2\beta H^{\mathrm{FP}}(0) - 2\beta\sigma_u^{\mathrm{PM}}(t)D + (t+1)\frac{d\sigma_u^{\mathrm{PM}}(t)}{dt}D\right). \quad (20)$$

Hence, we cannot guarantee the monotonic behavior of the second moment matrix $H^{\mathrm{FP}}(\tau)$ unlike the linear Fokker-Planck equation. For example, if the initial density $p(x,0)$ is not on $\mathcal{M}$ but has the common second moments with the equilibrium density, we cannot expect the right-hand side of (20) is zero and the second moment matrix possibly oscillates around its equilibrium.

3.2. Convergence rate of the solution of the PME to $\mathcal{M}$

We use the result [14, 17] that a solution of the NFPE decays exponentially with respect to the divergence, i.e.,

$$\mathcal{D}[p(x,\tau)\|p_\infty(x)] \le \mathcal{D}[p(x,0)\|p_\infty(x)]e^{-2\beta\tau}. \quad (21)$$

Proposition 5 *Let $u(x,t)$ be a solution of the PME and $\hat{u}(x,t)$ be the m-projection of $u(x,t)$ to the q-Gaussian family $\mathcal{M}$ at each t. Then $u(x,t)$ asymptotically approaches to $\mathcal{M}$ with*

$$\mathcal{D}[u(x,t)\|\hat{u}(x,t)] \le \frac{C_0}{1+t},$$

where C_0 is a constant depending on the initial function $u(x,0)$.

By the Csiszar-Kullback inequality [14] we can also conclude the L^1 convergence of $u(x,t)$ to $\mathcal{M}$ with the same rate.

4. Conclusions

We show that information geometric concepts on the manifold of the q-Gaussian densities provides us with a novel point of views to the behavioral study of solutions for the PME or NFPE. Geometric characterization of the self-similar solution [27, 28] will be discussed elsewhere.

References

[1] M. Muskat, McGraw-Hill, New York, (1937).
[2] J. Buckmaster, J. Fluid Mech. 81, 735 (1977).
[3] E. W. Larsen and G. C. Pomraning, SIAM J. Appl. Math. 39, 201-212 (1980).
[4] W. L. Kath, Physica D, 12, 375-381 (1985).
[5] G. I. Barenblatt, Cambridge Univ. Press, Cambridge, (1996).
[6] A. Friedman and S. Kamin, Trans. Amer. Math. Soc. 262, 551-563 (1980).
[7] D. G. Aronson, Lecture Notes in Mathematics 1224. Springer-Verlag, Berlin/New York, (1985).
[8] W. Newman, J. of Math. Phys. 25, 3124-3127 (1984).
[9] J. Ralston, J. of Math. Phys. 25, 3120-3123 (1984).
[10] A. R. Plastino and A. Plastino, Physica A 222, 347 (1995).
[11] M. Shiino, J. Phys. Soc. Jpn. **67**, 3658 (1998).
[12] T. D. Frank, Physics Letters A 290, 93-100 (2001).
[13] T. D. Frank, Physica A 310, 397-412 (2002).
[14] J. A. Carrillo and G. Toscani, Indiana Univ. Math. J. 49, 113-141 (2000).
[15] F. Otto, Comm. Partial Differential Equations 26, 101-174 (2001).
[16] J. L. Vázquez, J. Evol. Equ. 3, 67-118 (2003).
[17] G. Toscani, J. Evol. Equ. 5:185-203 (2005).
[18] C. Tsallis, S. V. F. Levy, A. M. C. Souza and R. Maynard, Physical Review Letters, 75, 3589-3593 (1995), Errata, 77, 5442 (1996).
[19] S-I. Amari, *Differential-Geometrical Methods in Statistics*, *Lecture Notes in Statistics*, Vol. 28, Springer-Verlag, 1985.
[20] S-I. Amari and H. Nagaoka, Trans. Math. Mono., Vol.191, AMS (2000).
[21] S. Eguchi, Sugaku Expositions, Vol. 19, No. 2, 197-216 (2006), (originally *Sūgaku*, 56, No.4, 380-399 (2004) *in Japanese*.)
[22] N. Murata, T. Takenouchi, T. Kanamori and S. Eguchi, Neural Computation, 16(7), 1437-1481 (2004).
[23] J. Naudts, Physica A 316, 323-334 (2002).
[24] J. Naudts, Physica A 340, 32-40 (2004).
[25] T. Wada and A.M.Scarfone, Physics Letters A, 335, 351-362 (2005).
[26] H. B. Callen, Thermodynamics and an Introduction to Thermostatistics, second ed., Wiley, New York (1985).
[27] G. I. Barenblatt, Prikl. Mat. Mekh. 16, 67-78(1952) (in Russian).
[28] R. E. Pattle, Quart. Jour. Mech. Appl. Math. 12, 407-409 (1959).