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Robust and Non-Robust ω-limit Orbits in 1D Cellular Automata

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Abstract—The two most popular classifications of 1D elementary Cellular Automata rules are based on the dynamics of the so-called *robust* ω-limit orbits, which can be observed when a long random bit string is used as initial state. In this paper, we introduce a classification that takes into account also the dynamics of the *non-robust* ω-limit orbits, which can be observed only for very specific initial states.

1. Introduction

Cellular automata (CA) are a classical example of how interdependent elementary structures can generate complex dynamics. In the last few years, this topic has been successfully addressed by using the theory of Nonlinear Dynamics [1], which considers CA as a special case of Cellular Nonlinear Networks (CNNs). One of the main results of such approach has been the introduction of a new classification of CA rules that is based on the notion of ω-limit orbit, which we will introduce later in this paper. Such classification is based on *quantitative* criteria, and it differs from the popular one proposed by Wolfram [2] which rather considers *qualitative* criteria. Still, both Chua's and Wolfram's classifications are based on the so-called *robust* ω-limit orbits while they ignore the dynamics of so-called *non-robust* ω-limit orbits. In this paper, we propose yet another classification of the CA local rules, but in this case based on the properties of both the *robust* and the *non-robust* ω-limit orbits.

The paper is structured as follows: in Sec 2, we introduce some fundamental concepts of Cellular Automata; in Sec. 3, we discuss both Chua's and Wolfram's classifications; in Secs. 4 and 5, we present several examples concerning the robust and the non-robust ω-limit orbits, respectively; in Sec. 6, we draw the conclusions.

2. Brief notes on Cellular Automata

Cellular Automata consist of regular uniform lattice of cells assuming a finite number of states; here, we consider one-dimensional CA in which cells are arranged in an array of length $L = I + 1$ and can take only two states: 0 and 1. For instance, a bit string $\mathbf{x}$ at the generic time step n is

$$\mathbf{x}^n = (x_0^n x_1^n \dots x_{I-1}^n x_I^n) \quad (1)$$

where the subscript indicates the position of the cell in the array. Hereafter, letters in bold indicate bits strings, and letters in italics are used for the single bits.

Cells are updated synchronously and the time evolution of a bit string can be effectively summarized by the notation

$$\mathbf{x}^{n+1} = f(\mathbf{x}^n) \quad (2)$$

in which the superscript indicates the iteration. The state of each cell at iteration $n + 1$ depends on the states of its neighbors (here we consider only the nearest neighbors) at iteration n:

$$x_i^{n+1} = f(x_{i-1}^n x_i^n x_{i+1}^n). \quad (3)$$

In the following, we use periodic boundary conditions, which means that

$$x_0^{n+1} = f(x_I^n x_0^n x_1^n) \quad \text{and} \quad x_I^{n+1} = f(x_{I-1}^n x_I^n x_0^n) \quad (4)$$

Under the restrictions detailed above, there are only 256 possible functions f, called rules, which we can be denoted by f_0 up to f_{255}. For instance, the notation:

$$\mathbf{x}^{n+2} = f_{110}(f_{110}(\mathbf{x}^n)) \quad (5)$$

indicates the application of rule 110 to the bit string $\mathbf{x}^n$ two times to obtain the bit string $\mathbf{x}^{n+2}$.

If the functions f_i are deterministic and the length L of the bit string is finite, then the evolution of an arbitrary initial state under an arbitrary rule f_N will end up in a periodic orbit, in the sense that there exist p and T such that

$$\mathbf{x}^p = \mathbf{x}^{p+T} \quad (6)$$

Obviously, $\mathbf{x}^{p'} = \mathbf{x}^{p'+T}$, for all $p' > p$. The bit strings from $\mathbf{x}^0$ to $\mathbf{x}^{p-1}$ are said to belong to the *transient*, which has length p, while the bit strings from $\mathbf{x}^p$ on are said to belong to the *periodic orbit*, which has length T. In this paper, we have opted for the terminology ω-limit orbits borrowed from classical nonlinear dynamical systems [3]. Even though the classical usage of the word *orbit* allows it to be the entire trajectory (both transient and steady state), the definition of ω-limit orbit requires it to be periodic for finite L. For this reason, in our terminology the ω-limit orbit excludes the transient part of the orbit and, for finite L, it coincides with the periodic orbit.

For some rules, it may happen that the there exists a τ for which:

$$\mathbf{x}^p = \mathcal{S}^{\sigma}(\mathbf{x}^{p+\tau}) \quad (7)$$

where $\mathcal{S}^\sigma$ indicates a shift, left or right, by σ positions, where conventionally σ is positive for left shifts and negative otherwise. Therefore, an ω-limit orbit can be characterized by its parameters $\tau > 0$ and $|\sigma| \geq 0$.
Since it is unfeasible to analyze the evolution of CA rules starting from all 2^L possible initial states when L is big, researchers use long (e.g., L greater than 100) random initial bit strings to characterize the behavior of a rule, as explained in detail in Sec. 3. The ω-limit orbits found via this procedure are said to be *robust* because they can be observed starting from a generic initial state. Rules can also have some ω-limit orbits, called *non-robust*, that can be reached only from some very specific initial states.

3. Classification of CA rules by Wolfram and Chua

3.1. Wolfram's classification

Cellular Automata local rules can be grouped according to many criteria, and several different classifications have been presented so far [4] [5] [6] [7] [8] [9] [10] [11]. Probably, the most famous is due to Wolfram [2], who proposed to classify CA local rules into four classes (here labeled from 'W1' to 'W4'), depending on the evolution of the system from a random initial state:

- W1: evolution leads to a homogeneous state;
- W2: evolution leads to a set of separated simple stable or periodic structures;
- W3: evolution leads to a chaotic pattern;
- W4: evolution leads to complex localized structures, sometimes long-lived.

On the one hand, this classification can be applied to any Cellular Automaton model, regardless the number of states, spatial arrangement, neighborhood etc.; on the other hand, it has received criticism for being based on empirical criteria (e.g., see [12]).

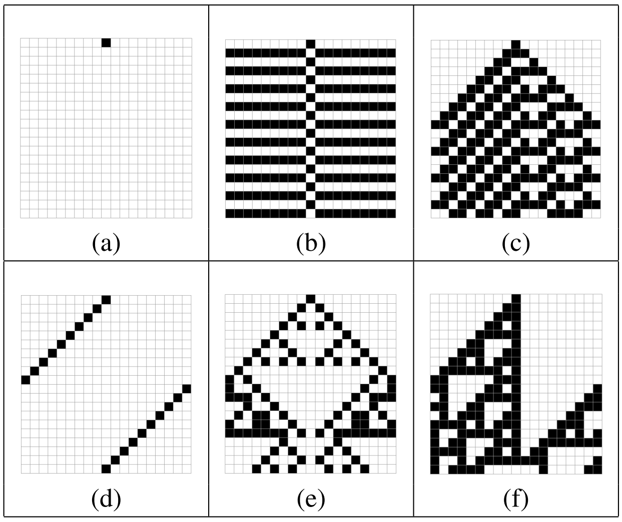
3.2. Chua's classification

An alternative classification scheme composed by six different groups (labeled from 'C1' to 'C6') was proposed by Chua in [1]. In this case, the feature used to discriminate the rules is the *robust* behavior of the ω-limit orbits found by using random bit strings:

- C1: rules exhibiting robust period-1 ω-limit orbits;
- C2: rules exhibiting robust period-2 ω-limit orbits;
- C3: rules exhibiting robust period-3 or period-6 ω-limit orbits;
- C4: rules exhibiting robust σ_τ-shift ω-limit orbits, where σ and τ do not depend on the initial bit string or the length L;
- C5: bilateral local rules exhibiting a robust σ_τ-shift ω-limit orbits, where σ and τ depend on the initial bit string and/or the length L;
- C6: non-bilateral local rules exhibiting a robust σ_τ-shift ω-limit orbits, where σ and τ depend on the initial bit string and/or the length L.

Some examples of spatial-temporal patterns generated by the rules belonging to the different groups are displayed in Table 1.

Table 1: Examples of spatial-temporal patterns obtained from a single black pixel for six different rules and the classifications according to Wolfram (W) and Chua (C), as described in Sec. 3: (a) Rule 0, W1 and C1; (b) Rule 51, W2 and C2; (c) Rule 62, W2 and C3; (d) Rule 170, W2 and C4; (e) Rule 90, W3 and C5; (f) Rule 110, W4 and C6.



3.3. Relationship between Wolfram's and Chua's classifications

These two classifications are related to each other: rules belonging to W1 are a proper subset of those of C1; rules belonging to W2 can be in any of the Chua's groups from C1 to C4; finally, rules belonging to W3 and W4, can be either in C5 or in C6, depending on their characteristics. Therefore, we can summarize these results as follows:

$$W1 \subset C1$$
$$W2 \equiv ((C1 \backslash W1) \cup C2 \cup C3 \cup C4)$$
$$(W3 \cup W4) \equiv (C5 \cup C6)$$

We ought to emphasize that Chua's classification has been expressly developed for 1D elementary CA, and hence it could be different for other models (e.g., more states, different neighborhood), while Wolfram's classification can be applied without changes independently of the particular

CA model under consideration.
Obviously, the presence of non-robust ω-limit orbits is peculiar to rules of W1 and W2, in Wolfram's classification, and of C1, C2, C3, and C4, in Chua's classification, since the classes W3, W4, C5, and C6 are characterized by the fact of not having a dominant, i.e., robust, kind of orbit.

4. Results on the robust ω-limit orbits

An exhaustive analysis of all CA rules has led to the conclusion that all globally-independent local rules have *at most* two robust ω-limit orbits. In particular, the rules of C5 and C6 do not have any robust ω-limit orbit, by definition; rules 14, 43, 57, 142, 184 (all of them belonging to C4) have two robust ω-limit orbits; all other rules have only one robust ω-limit orbit. The rules of the first four groups in Chua's classification can be grouped according to their values of the Bernoulli parameters σ and τ, as shown in Table 2. In particular, the rules of C1, C2, and C3 are in the first column, because they are periodic in time but not in space and hence $\sigma = 0$, while the rules of C4 (so-called Bernoulli rules) have $\sigma \neq 0$ since they are periodic in space *and* time.

Table 2: Distribution of the rules of the first four classes in Chua's classification according to the values of σ and τ of their robust ω-limit orbits. All these rules belong either to W1 or to W2 in Wolfram's classification.

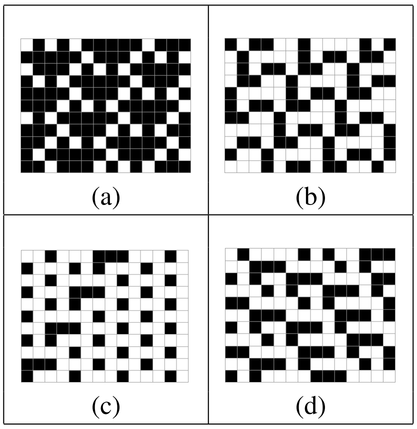
	$\sigma = 0$	$\lvert\sigma\rvert = 1$	$\lvert\sigma\rvert = 2$
$\tau = 1$	C1	C4	
$\tau = 2$	C2	C4	C4
$\tau = 3$ and $\tau = 6$	C3		

Observe that no Bernoulli rule has robust ω-limit orbits with $\tau > 2$. Also, no rule can have robust ω-limit orbits with $\lvert\sigma\rvert > \tau$ because this would imply that the information travels in space faster than one bit per iteration, which is not possible in our nearest-neighbors model. Nevertheless, in the examples displayed in Table 3 (a), (b), and (d) the information *apparently* travels faster than one bit per iteration, but this effect is due to particular spatial configurations that can occur only in non-robust periodic orbits.

5. Results on the non-robust ω-limit orbits

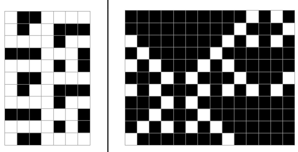
Both Wolfram's and Chua's classification are based on the results obtained by using one or more *long* random bit strings as initial state, but this method allows us observe only the *robust* ω-limit orbits, as explained in Sec. 2. However, as already pointed out in [13, 14] there exist rules whose robust ω-limit orbits are 'dull', while their non-robust ω-limit orbits exhibit a variety of interesting dynamics.
For example, Rule 164 is W2 and C1, and hence they are not considered 'interesting' in such classifications. A similar situation happens for Rule 37 which is W2 and C2. However, both these rules have a non-robust ω-limit orbit with $\tau = 3$ and $\sigma = 1$ for L=14, as it can be observed in Table 3. Other two examples are given by Rules 9 and 25, both of them W2 and C4, which have non-robust ω-limit orbit with $\sigma = 2$ and $\sigma = 3$, respectively, and $\tau = 3$.

Table 3: Examples of non-robust ω-limit orbits with $\tau = 3$: (a) Rule 164, W2 and C1 ($\sigma = 1$); (b) Rule 37, W2 and C2 ($\sigma = 1$); (c) Rule 9, W2 and C4 ($\sigma = 2$); (d) Rule 25, W2 and C4 ($\sigma = 3$).



Many of the 70 globally-independent rules belonging to the first two Wolfram's groups (or, equivalently, to the first four Chua's group) have only one kind of orbit. For example, all ω-limit orbits of Rule 76 are period-1, and all ω-limit orbits of Rule 162 have $\tau = 1$ and $\sigma = 1$. However, this is not always the case: some rules have orbits with both $\tau = 1$ and $\tau = 2$. Remarkably, there are a few rules – namely, 9, 25, 37, 74, 94, 164 – having non-robust orbits with $\tau = 3$. The reason why such a feature is so interesting is that a classical work [15] in Nonlinear Dynamics proofs that in a continuous function defined over an interval, the presence of a period-3 orbit implies the presence of orbits with any other period. Obviously, this result does not apply directly to Cellular Automata because, in general, they describe discontinuous functions. Nevertheless, we noticed that all rules in the last two groups of Chua's classification have at least one periodic orbit with $\tau = 3$ and, at the same time, the five rules with $\tau = 3$ mentioned above tend to have a more complex behavior than the remaining of the first four groups of Chua's classification. For instance, in Table 4 orbits with $\tau > 3$ for two of the five rules aforementioned are displayed. Therefore, somehow the parameter $\tau = 3$ may give an indication of richer dynamics, as also confirmed by the analysis made by using other methodologies [16].
For this reason, besides the classification of rules according to their robust orbits, as done by Wolfram and Chua, we suggest a classification based on their non-robust or-

Table 4: Examples of non-robust ω-limit orbits with τ greater than 3: (a) Rule 9 , with $\tau = 5$ and $\sigma = 0$; (b) Rule 164, with $\tau = 7$ and $\sigma = 7$.



(a) (b)

bits. In particular, we can distinguish three classes: i) rules with only robust ω-limit orbits; ii) rules whose non-robust orbits have $\tau = 1$ and/or $\tau = 2$; iii) rules whose non-robust orbits have $\tau \geq 3$. This last group contains at least one rule from each of the first four Chua's groups.

6. Conclusion

In this paper, we propose a classification of the CA local rules based on the properties of both *robust* and *non-robust* ω-limit orbits. Thanks to this new approach, we found that some rules belonging to C1, C2, C3, and C4 – or, equivalently, to W1 and W2 – have at least some ω-limit orbits with characteristics similar to those of the rules belonging to C5 and C6 – or, equivalently, to W3 and W4. This result opens a new scenario to analyze the computational properties of Cellular Automata: some rules that have been considered 'uninteresting' so far become suddenly rich in dynamic behaviors, still to be studied.

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