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Using permutation complexity tools to analyze complex spatiotemporal dynamics

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Abstract—In recent years the tools of permutation complexity have found interesting applications in time series analysis. These tools make use of any quantity or functional based on the order relations (permutations) appearing between consecutive elements of a time series. We propose to extend the use of permutation complexity tools to the analysis of complex spatiotemporal dynamics. In this paper we illustrate how interesting properties of a paradigmatic class of models of spatiotemporal dynamics, the Cellular Automata, in particular its topological entropy, can be estimated using permutation complexity tools. We discuss later the implications of this result and how similar ideas can be applied to more general types of spatiotemporal data.

1. Introduction

If the state space of a dynamical system is equipped with a total ordering, this additional structure can be taken into account when analyzing its behavior. The result is what we call permutation analysis, an approach to dynamics and dynamical complexity characterized by conceptual simplicity, an algebraic flavor and computational speed. The tools of permutation analysis include ordinal patterns, order-isomorphy, metric and topological permutation entropy, discrete entropy, and regularity parameters. Permutation analysis has been successfully applied to the estimation of entropies [1, 2], measure of complexity in time series [3], recovery of control parameters of unimodal maps from symbolic sequences [4], characterization of synchronization [5], detection of determinism in time series [6, 7], etc. The next challenge is to extend these applications to physical systems, and more specifically to space-time systems.

In a recent work [8] we proposed a way to apply the tools of permutation analysis to the study of complex spatiotemporal systems. The aim of this paper is to illustrate some of the aspects of these ideas making use of Cellular Automata (CA) as simple models of spatially extended physical systems. Cellular automata were introduced by Ulam [9] and von Neumann [10], and are currently the object of intensive study in mathematical physics, computer science, biology, etc. [11, 12, 13]. For a readable account on cellular automata and their remarkable performance in physical modeling (including turbulence, space-time chaos, symmetry-breaking, and ordering), see, e.g. [14]. The ideas exposed here can later be adapted to more general spatiotemporal data sets by discretizing them both in space and in time, for a detailed description we refer to the reader to Refs. [8, 15].

2. Cellular automata as dynamical systems

For our purposes, it suffices to consider only one-dimensional CA. In this case, the configuration space is the two-sided sequence space

$$S^{\mathbb{Z}} = \{(s_n)_{n\in\mathbb{Z}} = (..., s_{-k}, ..., s_{-1}, s_0, s_1, ..., s_k, ...) : s_n \in S\}. \quad (1)$$

The state of cell i at time $t \geq 0$ will be denoted $s_t(i)$. At each time step $t+1$, the previous state at each cell i, $s_t(i) \in S$, is updated according to the *local rule* $f : S^{2l+1} \to S$ of the form

$$s_{t+1}(i) = f(s_t(i-l), s_t(i-l+1), ..., s_t(i+l)). \quad (2)$$

The local rule f leads to a *global transition map* of the configuration space, $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ defined in the obvious way:

$$\begin{aligned} F(..., s_t(i), ...) &= (..., f(s_t(i-l), s_t(i-l+1), ..., s_t(i+l)), \quad (3)\\ &= (..., s_{t+1}(i), ...). \quad (4) \end{aligned}$$

We are going to deal here with finite size CA. The state vector of the system at time t will be denoted as $\mathbf{x}_t^+$, so

$$\mathbf{x}_t^+ = (s_t(1), s_t(2), ..., s_t(N)), \quad (5)$$

where N is the length of the CA.

The topological entropy of a dynamical system provides a good estimation of its complexity. For cellular automata, one can use the following procedure to estimate it [16]. Let $R(w,t)$ be the number of distinct rectangles of width w and height (temporal extent) t occurring in a space-time evolution diagram of $(S^{\mathbb{Z}}, F)$, Fig. 1. Then

$$h_{top}(F) = \lim_{w\to\infty} \lim_{t\to\infty} \frac{1}{t} \log R(w,t). \quad (6)$$

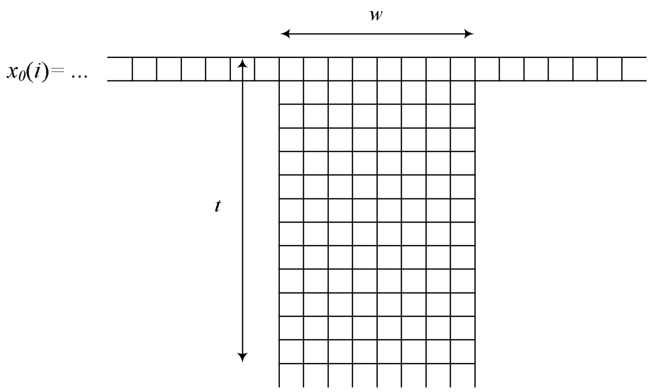

Figure 1: Geometrical illustration of the rectangles $R(w, t)$ used in Eq. (6).

Therefore, the complexity of the CA can be measured by the number of distinct words or patterns per time unit generated by the global transition map F as time evolves. It follows that

$$h_{top}(F) \leq 2l \log |S| , \quad (7)$$

where l is the neighborhood size of the automaton and $|S|$ is the cardinality of S.

In next section we are going to describe some tools of permutation analysis and in particular the notion of ordinal pattern. After this, we are going to discuss how they can be used to estimate the topological entropy of a CA.

3. Ordinal patterns and topological entropy

Let $x_0^\infty = (x_n)_{n\in\mathbb{N}_0}$ be a sequence generated by a source $\mathbf{X}$ whose elements x_n belong to a space endowed with a total ordering $<$. We say that a length-L block (segment, word,...) $x_n^{n+L-1} = x_n, x_{n+1}, ..., x_{n+L-1}$ defines the *ordinal* (L-) *pattern* $\pi = \langle \pi_0, ..., \pi_{L-1} \rangle$ if

$$x_{n+\pi_0} < x_{n+\pi_1} < ... < x_{n+\pi_{L-1}}, \quad (8)$$

where in case $x_i = x_j$ and $i < j$, we set $x_i < x_j$ for definiteness. Note that $\pi_0, ..., \pi_{L-1}$ is a permutation of the numbers $0, 1, ..., L-1$; for this reason, ordinal patterns are sometimes called permutations too. The set of ordinal L-patterns will be denoted by $\mathcal{S}_L$.

Topological permutation entropy (otherwise called the capacity of the source $\mathbf{X}$), is defined as

$$h^*_{top}(\mathbf{X}) = \lim_{L\to\infty} h^*_{top}(x_0^{L-1}) = -\lim_{L\to\infty} \frac{1}{L} \log N(L), \quad (9)$$

where $N(L)$ is the number of allowed ordinal L-patterns in the 'messages' output by $\mathbf{X}$. Thus, the estimation of $h^*_{top}(\mathbf{X})$ boils down to counting the number of distinct patterns in sliding windows of size L. For a wide class of dynamical systems (maps), the topological permutation entropy is equal to the topological entropy [15].

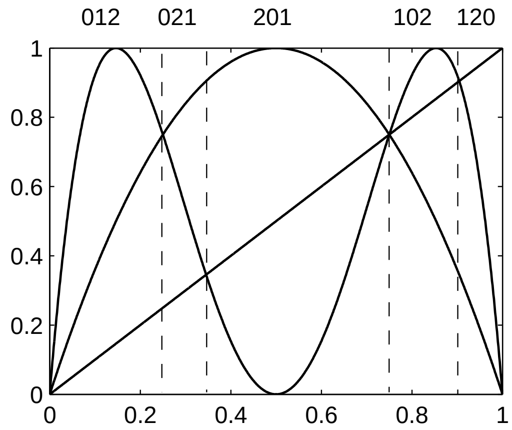

Figure 2: Graph of the logistic map $x_{n+1} = g(x_n)$, of its second itereate and of the graph $x_{n+1} = x_n$. Clearly there is no point in the $[0, 1]$ interval for which $g^2(x) < g(x) < x$ so the pattern $\langle 2, 1, 0\rangle$ is forbidden.

For a map $f : I \to I$ we say that an ordinal L-pattern π is *allowed* or *admissible* for T if there exists $x \in I$ of type π, otherwise the ordinal pattern is *forbidden* for T. Since $|\{\pi \in \mathcal{S}_L\}| = L!$ and $\lim_{L\to\infty}(\log L!/L) = \infty$, it follows from the result cited above that that orbits of quite general maps have necessarily forbidden patterns [17, 18]. The forbidden ordinal patterns of the shift and signed shift transformations on sequence spaces have been studied in [19] and [20], respectively.

In order to see this, consider sequences generated using the logistic map $x_{n+1} = g(x_n) = 4x_n(1 - x_n)$. In Fig. 3 we can see how for this system there are forbidden ordinal patterns, in particular the pattern $\langle 2, 1, 0\rangle$ cannot be observed on a time series of this system. On the other hand, unconstrained random sequences have no forbidden patterns with probability one. This being the case, the existence of forbidden patterns (together with the robustness of ordinal patterns to additive noise) can be exploited to discriminate deterministic noisy time series from *white noise* (an independent and identically distributed random process), with a remarkable success [7].

4. Topological entropy of one-dimensional CA

Let $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ be the global transition map of an elementary CA. We consider for simplicity here CA where $S = \{0, 1\}$. As mentioned in Sec. 2, its dynamical complexity can be measured by means of the topological entropy (6).

Another possibility consists in using the topological permutation entropy $h^*_{top}(F)$ instead, that can be computed as follows.

The topological permutation entropy of the automaton defined by the local rule (2), can be estimated via the ordinal patterns of its global map $F : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ using two equivalent procedures. The first one would be to estimate the topological permutation entropy of the "sequence" of spatiotemporal data $\{\mathbf{x}_t^+\}_{t=1}^M$, using for this purpose lexicographical order, that is, we say that $\mathbf{x}_m^+ < \mathbf{x}_n^+$ if $s_m(i) = s_n(i)$ for $1 \le i \le j-1$ and $s_m(j) < s_n(j)$. An equivalent way to achieve the same goal would be to consider the sequence of real numbers $\{\phi_t^+\}_{t=1}^M$, that for $S = \{0, 1\}$ has the form

$$\phi_t^+ = \phi(\mathbf{x}_t^+) = \sum_{i=1}^{N} \frac{s_t(i)}{2^i} \in [0, 1). \qquad (10)$$

Remarkably, and similarly to what happened with maps, the topological permutation entropy of the CA is equal to its topological entropy [15]. This result provides a first insight on the validity of the permutation complexity tools for the study of spatiotemporal dynamics. We provide now some numerical evidences validating this result.

5. Numerical results

In order to illustrate our ideas we consider the CA with local rule

$$f(p, q, r) = p + r \bmod 2, \qquad (11)$$

which is an instance of a positively expansive CA, thus with complicated dynamics. The topological entropy for this CA is $h_{top}(F) = 2\log 2 = 2$ bit/iteration [16].

Figure 3 shows different aspects of the cellular automaton, fixing the size $N = 250$. We can see in Fig.3 (a) the time evolution of cells $1 \le i \le 250$, which clearly displays a complex spatiotemporal dynamics. In Fig. 3 (b) we can see a plot of the sequence $\{\phi_t^+\}_{t=1}^M$ for this CA, it also has a corresponding complex appearance. Notice that simple behaviour on the CA would typically yield to a simple appearance in the corresponding sequence of ϕ_t^+. The existence of certain structure in this seemingly complex sequence is revealed in Fig. 3 (c) where the "return map" of the sequence, i.e. the plot of $\phi^+(\mathbf{x}_t^+)$ vs $\phi^+(\mathbf{x}_{t+1}^+)$ is shown. This graph has seemingly a fractal structure; if the sequence $\{\phi_t^+\}_{t=1}^M$ were purely random one would expect to see a random cloud of points.

Figure 3 (d) illustrates that our result is verified for this CA, as claimed. There we show the convergence of the topological permutation entropy rates of order L computed using the ideas provided above to the value of $h_{top}(F) = 2$ bit/symbol. This is a simple example of how the tools of permutation analysis can be used in order to quantify the complexity of a CA.

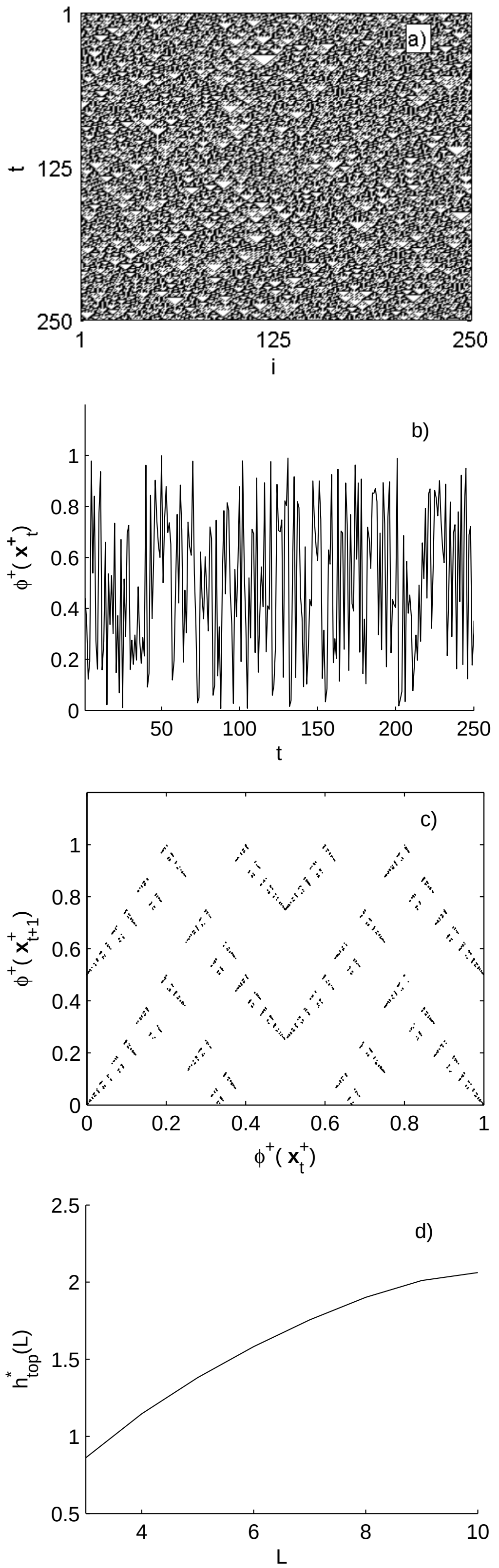

Figure 3: Different aspects of a positively expansive CA (see text). Plot (d) shows the convergence of the topological permutation entropy of the automaton to its topological entropy.

6. Conclusion and outlook

In this paper we have provided a simple example on how the tools of permutation analysis can be used to estimate the complexity of a system with complex spatiotemporal dynamics, using as an example a CA. We have shown that for a simple class of CA it is possible to estimate its topological permutation entropy and that it converges to its topological entropy.

This result illustrates some of the basic features of the application of permutation analysis to spatiotemporal dynamics. In a recent work we have shown how these ideas can be extended to the study of spatiotemporal data of Coupled Map Lattices (CMLs) and even to real spatiotemporal data from magnetoencephalograms (MEGs) [8]. Note that the state of each site of these two type of data are not (in principle) discrete. However, in the examples studied we have notice that it suffices to discretize the state of each site using two simbols (for example fixing $s_t(i) = 0$ if the state of the site at time t is below its mean value and $s_t(i) = 1$ if it is above its mean value). Numerical evidence show that a permutation analysis of the resulting discretized spatiotemporal data provides a way to quantify the complexity of the systems considered. Furthermore, it can be shown that a combination of the permutation analysis in time, using the above ideas, and in space, considering the state vector as a sequence, allows one to distinguish quite neatly between different types of complex spatiotemporal data. All these evidences, together with the computational speed of the permutation complexity methods and its robustness against noise (due to the fact that they rely on inequalities) makes the permutation complexity analysis a promising tool for the analysis of complex spatiotemporal data.

Acknowledgments

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