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Iterative Linear Quadratic Regulator Using Variational Equation: A Case Study with Swing-up Control of Inverted Pendulum

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Abstract— This paper discusses the discretization method that is used in the implementation of an optimization-based nonlinear feedback controller called Iterative Linear Quadratic Regulator (ILQR). While a finite difference approximation is usually used in the field of control engineering, we propose to use the variational equation in the procedure of ILQR. The effectiveness of using the variational equation is examined through a case study with swing-up control of an inverted pendulum.

1. Introduction

Optimal control is a fundamental problem in modern control theory, and the Linear Quadratic Regulator (LQR) method has been established for linear systems. However, it is difficult to solve a nonlinear optimal control problem in many cases. Recently, a method called Iterative LQR (ILQR)[1] is often used not only for trajectory optimization but also for feedback control (as a model predictive control method), and its effectiveness is demonstrated in some applications [2, 3]. In ILQR, the nonlinear state equation is iteratively linearized around a nominal trajectory, and the resultant LQR problem is solved at each step. When implementing ILQR, in the field of control engineering, a finite difference approximation, such as Euler method, is generally used to discretize the original continuous-time model, see e.g. [4]. However, the approximation error increases as the increase in the sampling period. Therefore, it would be appropriate to use variational equation [5], which enables direct linearization of the transition map of the discretized system and is commonly used for analysis of nonlinear dynamical systems.

In this paper, the two ILQR methods using the finite difference approximation and the variational equation are applied to the swing-up control of an inverted pendulum, and the effectiveness of the variational equation is examined. To clearly evaluate the optimality of these trajectories, the exact solution to the Hamilton-Jacobi equation for the optimal control problem is also derived by the stable manifold method [6, 7] and compared with the results of ILQR.

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2. ILQR Using Variational Equation

In this section, we first introduce the framework of ILQR for discrete-time systems proposed in [1], and then describe how the variational equation can be applied to derive the time-varying linear system used in ILQR.

2.1. ILQR for Discrete-time Systems

Consider the discrete-time control system described by the state equation:

$$x(k+1) = \bar{f}(x(k), u(k), k) \tag{1}$$

where x stands for the state variable, u for the control input, and $\bar{f}(x(k), u(k), k)$ is the function representing the time evolution of the state. ILQR is an iterative method, and each iteration starts with a series of nominal control inputs $\{\bar{u}_k\}_{k=0}^{N-1}$ calculated in the previous step $k-1$, and the nominal state trajectory $\{\bar{x}_k\}_{k=0}^{N}$ obtained by applying these inputs to the controlled system. Let $\delta x_k := x_k - \bar{x}_k$ and $\delta u_k := u_k - \bar{u}_k$ be the deviations from the nominal trajectory and input, respectively, then the system (1) can be linearized as

$$\delta x_{k+1} = A_k \delta x_k + B_k \delta u_k \tag{2}$$

where $A_k := \partial \bar{f}/\partial x_k$ and $B_k := \partial \bar{f}/\partial u_k$. Then, for this linear model (2), we solve the LQR problem with the following cost function at each step k:

$$J = (\bar{x}_N + \delta x_N)^\top Q_f (\bar{x}_N + \delta x_N) + \sum_{k=0}^{N-1} \left\{ (\bar{x}_k + \delta x_k)^\top Q (\bar{x}_k + \delta x_k) + (\bar{u}_k + \delta u_k)^\top R (\bar{u}_k + \delta u_k) \right\} \tag{3}$$

where $Q_{\mathrm{f}} \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ are semidefinite matrices and $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix.

2.2. Linearization by Variational Equation

The matrices A_k and B_k in Eq. (2) are defined as the Jacobians of $\bar{f}$. However, if the system is given as a continuous-time system $\dot{x} = f(x, u, t)$, no analytic expression is available for $\bar{f}$. In the field of control engineering, a finite difference approximation, such as Euler method, is generally used, see e.g. [4], and the system (1) is approximated by

$$x(t+\Delta t) \simeq x(t) + \Delta t f(x,u). \tag{4}$$

Thus, the matrices A_k and B_k are given by

$$\begin{aligned} A_k &= \frac{\partial \bar{f}}{\partial x_k} = I + \Delta t \frac{\partial f}{\partial x_k}, \\ B_k &= \frac{\partial \bar{f}}{\partial u_k} = \Delta t \frac{\partial f}{\partial u_k}. \end{aligned} \tag{5}$$

However, with the finite difference method, the approximation accuracy deteriorates as the increase in the sampling period Δt. In this paper, we propose to apply the variational equation [5] for deriving the time-varying linear system used in ILQR. For this, from the state x_k and the input u_k, the next state x_{k+1} is given by solving the differential equation from time t_k to $t_k + \Delta t$:

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}, t), \quad \tilde{x}(t_k) = \tilde{x}_k := \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{6}$$

where $\tilde{x} = [x^\top \ u^\top]^\top$ and $\tilde{f} = [f(x,u)^\top \ 0]^\top$. Let $\phi_t(\tilde{x}_k, t_k)$ be the solution starting from the initial value $\tilde{x}_k$ of Eq. (6) at time t_k, then the following equations hold:

$$\dot{\phi}_t(\tilde{x}_k, t_k) = \tilde{f}(\phi_t(\tilde{x}_k, t_k), t), \tag{7}$$

$$\phi_{t_k}(\tilde{x}_k, t_k) = \tilde{x}_k. \tag{8}$$

Differentiating these by $\tilde{x}_k$ we obtain,

$$\frac{\partial \dot{\phi}_t}{\partial \tilde{x}_k}(\tilde{x}_k, t_k) = \frac{\partial \tilde{f}}{\partial \tilde{x}}(\phi_t(\tilde{x}_k, t_k), t)\frac{\partial \phi_t}{\partial \tilde{x}_k}(\tilde{x}_k, t_k), \tag{9}$$

$$\frac{\partial \phi_{t_k}}{\partial \tilde{x}_k}(\tilde{x}_k, t_k) = I. \tag{10}$$

Let $\Phi_t(\tilde{x}_k, t_k) = \partial\phi_t / \partial\tilde{x}_k(\tilde{x}_k, t_k)$, then Eqs. (7) and (10) can be rewritten as follows:

$$\dot{\Phi}_t(\tilde{x}_k, t_k) = \frac{\partial \tilde{f}}{\partial \tilde{x}}(\phi_t(\tilde{x}_k, t_k), t)\Phi_t, \tag{11}$$

$$\Phi_{t_k}(\tilde{x}_k, t_k) = I. \tag{12}$$

Here, $\Phi_{t_k+\Delta t}$ can also be described by

$$\Phi_{t_k+\Delta t} = \begin{bmatrix} \frac{\partial \bar{f}}{\partial x_k} & \frac{\partial \bar{f}}{\partial u_k} \\ 0 & 0 \end{bmatrix}. \tag{13}$$

Therefore, by solving Eqs. (7), (8), (11) and (12) simultaneously, we obtain A_k and B_k.

3. The stable manifold method

This section briefly reviews the stable manifold method [6] to calculate the exact optimal trajectory for the inverted pendulum problem discussed in the next section. Consider the control problem with the state equation and the cost function:

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0, \tag{14}$$

$$J = \int_0^\infty (x^\top Qx + u^\top Ru)\mathrm{d}t. \tag{15}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. For this control problem, the Hamilton-Jacobi equation is expressed as

$$H(x,\lambda) = \lambda^T f(x) - \frac{1}{4} g(x)R^{-1}g(x)^T\lambda + x^T Qx = 0. \tag{16}$$

The stabilizing solution of Eq. (16) is equivalent to the stable manifold of the origin of the following Hamiltonian system [9]:

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \lambda}(x,\lambda) \\ \dot{\lambda} = -\dfrac{\partial H}{\partial x}(x,\lambda) \end{cases} \tag{17}$$

Then, by introducing a suitable coordinate transformation T as

$$\begin{pmatrix} q \\ p \end{pmatrix} := T^{-1}\begin{pmatrix} x \\ \lambda \end{pmatrix}, \tag{18}$$

the system (17) can be rewritten as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A - \bar{R}\Gamma & 0 \\ 0 & -(A - \bar{R}\Gamma)^T \end{pmatrix}\begin{pmatrix} q' \\ p' \end{pmatrix} + \text{higher order terms} \tag{19}$$

where $\bar{R} = g(x)R^{-1}g(x)^\top$ and Γ is the stabilized solution of the Riccati equation consisting of linear terms of the form (16). Furthermore, by setting $F = A - \bar{R}\Gamma$ and letting n_s and n_u be higher-order nonlinear terms, Eq. (19) can be rewritten as follows:

$$\begin{cases} \dot{q} = Fq + n_s(t,q,p), \\ \dot{p} = -F^\top p + n_u(t,q,p). \end{cases} \tag{20}$$

We assume that F is an asymptotically stable $n \times n$ real matrix, and n_s, n_u are higher order terms with sufficient smoothness. We define the sequences $q_k(t,\xi)$ and $p_k(t,\xi)$ by

$$\begin{cases} q_{k+1} = \mathrm{e}^{Ft}\xi + \displaystyle\int_0^t \mathrm{e}^{F(t-s)} n_s(s, q_k(s), p_k(s))ds, \\ p_{k+1} = -\displaystyle\int_t^\infty \mathrm{e}^{-F^T(t-s)} n_u(s, q_k(s), p_k(s))ds. \end{cases} \tag{21}$$

for k=0,1,2, ..., and

$$\begin{cases} q_0 = \mathrm{e}^{Ft}\xi \\ p_0 = 0 \end{cases} \tag{22}$$

with arbitary $\xi \in \mathrm{R}^n$. Then the following theorem holds:

Theorem 1 *[6]. The sequences $q_k(t,\xi)$ and $p_k(t,\xi)$ are convergent to zero for sufficiently small $|\xi|$, that is, $q_k(t,\xi)$, $p_k(t,\xi) \to 0$ as $t \to \infty$ for all $k = 0, 1, 2, \ldots$ Furthermore, $q_k(t,\xi)$ and $p_k(t,\xi)$ are uniformly convergent to a solution of (20) on $[0,\infty)$ as $k \to \infty$ for sufficiently small $|\xi|$. Let $q(t,\xi)$ and $p(t,\xi)$ be the limits of $q_k(t,\xi)$ and $p_k(t,\xi)$, respectively. Then, $q(t,\xi)$, $p(t,\xi)$ are the solution on the stable manifold of (20), that is, $q(t,\xi)$, $p(t,\xi) \to 0$ as $t \to \infty$.*

For each k, the functions in (21) are calculated for the system (19) to obtain $p_k(t,\xi)$ and $q_k(t,\xi)$. Functions $x_k(t,\xi)$ and $\lambda(t,\xi)$ given by

$$\begin{pmatrix} x \\ \lambda \end{pmatrix} = T \begin{pmatrix} p \\ q \end{pmatrix} \tag{23}$$

form parameterizations of the approximate stable manifold. Since the obtained trajectories are solutions of the Hamilton-Jacobi equations, the computed trajectories are guaranteed to be optimal.

4. Swing-up Control of Inverted Pendulum

In this section, we present numerical simulations of the swing-up stabilization of an inverted pendulum by ILQR-based model predictive control, and compare it with the optimal solution obtained by the stable manifold method [6, 7]. The inverted pendulum system considered here is a twodimensional model, where the pendulum is attached on a massless cart that has been studied in [7]. The equation of motion of this inverted pendulum model is given by:

$$\dot{x} = f(x) + g(x)u \tag{24}$$

where $x := [x_1, x_2]^\top$. Here x_1 stands for the angle of the pendulum, x_2 for the angular velocity, and u for the acceleration of the cart. The functions f and g are given by

$$f(x) = \begin{pmatrix} x_2 \\ \dfrac{MGL\sin x_1 - ML^2 {x_2}^2 \sin x_1 \cos x_1}{J + ML^2 \sin^2 x_1} \end{pmatrix}, \tag{25}$$

$$g(x) = \begin{pmatrix} 0 \\ \dfrac{-L\cos x_1}{J + ML^2 \sin^2 x_1} \end{pmatrix}. \tag{26}$$

where M stands for the mass of the pendulum, G for the acceleration of gravity, L for the length of the pendulum to the center of gravity, and J for the moment of inertia. The setting of the parameter is the same as in [7].

The weights are set to $Q_f = 0$, $Q = \mathrm{diag}[2,\ 0.01]$, $R = 2$ in Eq.(3). The prediction horizon is fixed to $N = 10$, and the sampling period Δt is varied from 0.01s to 0.05s.

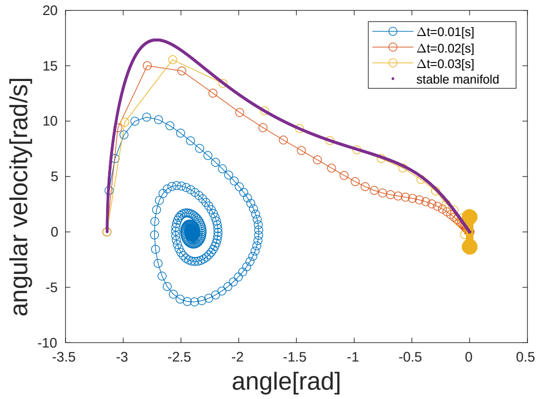
Figures 1a and 2a show the simulation results with the finite difference approximation and with the variational equation, respectively. In both two cases, when $\Delta t = 0.01$ s, the trajectory does not reach the upright position, $(x_1, x_2) = (0, 0)$, and converges in the middle of the swing up. When $\Delta t = 0.02$ s, it converges to $(x_1, x_2) = (0, 0)$, and when $\Delta t = 0.03$ s, the trajectory becomes closer to the optimal trajectory (shown by the purple line), suggesting that the longer prediction interval leads to better control performance. However, in Figure 1a, at $\Delta t = 0.03$ s, we can see that it oscillates without converging to $(x_1, x_2) = (0, 0)$. This is due to the increase in the approximation error of the finite difference method.

The effect of the approximation error becomes more pronounced for larger sampling periods as shown in Figure 1b. It can be seen that with the finite difference method, the state of the pendulum is not stabilized in the both cases of $\Delta t = 0.04$ and 0.05 s. In contrast, when the variational equation is used, as shown in Figure 2b, the pendulum can be stabilized at the upright position for both $\Delta t = 0.04$ and $\Delta t = 0.05$ s, while there is an overshoot passing through upright position when $\Delta t = 0.05$ s. This is because the larger the sampling period leads to the lower update frequency of the control input. Thus, there is a trade-off between the longer outlook horizon, which leads to a better control performance, and the lower update frequency of the input, which makes the feedback stabilization more difficult. Nevertheless, it can be seen that the overall control performance with the ILQR using the variational equation is better than that using the finite difference method due to low approximation error.

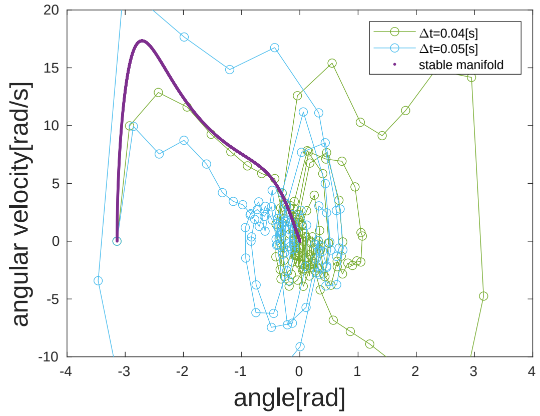
To achieve a better control performance, there are two ways to increase the outlook horizon: by increasing the prediction step size N and by enlarge the sampling period Δt, but increasing the step size N has the disadvantage of increasing the computation time to solve the resultant optimization problem. In the case of model predictive control, the optimization must be completed within the sampling period, and there is a limit on the step size N for real-time implementation. On the other hand, increasing the sampling period has the advantage that it does not directly affect the computational complexity. However, the finite difference method has a disadvantage that the approximation error increases when the sampling period is increased. The above simulation results clearly show that the use of variational equations can solve this drawback. This makes it possible to take a long outlook horizon, which leads to a better control performance, while it should be noted that too large sampling may deteriorate the control performance due to low update frequency of the control input.

5. Conclusion

In this paper, we discussed the discretization method that is used in ILQR. We simulate the swing-up stabilization of an inverted pendulum by ILQR with the finite difference approximation and with the variational equation. As a result, the effectiveness of the variational equation was confirmed by the fact that it was able to stabilize the swing-up even when the sampling period was large. In the future, we would like to verify the system with actual machines and

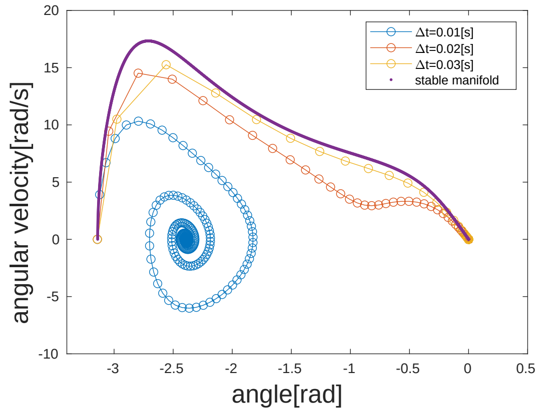


(a) Δt = 0.01 to 0.03 s

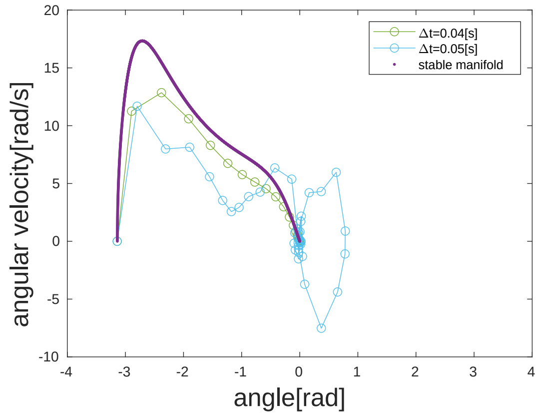


(b) Δt = 0.04 and 0.05 s

Figure 1: ILQR using finite difference method



(a) Δt = 0.01 to 0.03 s



(b) Δt = 0.04 and 0.05 s

Figure 2: ILQR using variational equation (proposed method)

with more complex controls.

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