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# Hyperjerk dynamics in hyperchaotic systems

Arturo Buscarino†, Luigi Fortuna†, Mattia Frasca†, and Antonio Gallo†

†DIEEI, University of Catania, Italy. Email: arturo.buscarino@dieei.unict.it

**Abstract**—In this paper the hyperjerk dynamics and the corresponding jerk functions for well known hyperchaotic circuits are presented. In particular, it is shown that the hyperjerk dynamics can not be obtained for each circuit variables, moreover it can not be obtained for all the variables of particular circuits.

## 1. Introduction

Chaos and hyperchaos are widely studied in the literature. Recently, particular attention has been devoted to the study on a particular class of such a type of circuits: the hyperjerk dynamical systems [1]. Our attention is focused in this paper to the hyperjerk with hyperchaotic dynamics. In fact, even if hyperjerk systems showing hyperchaotic dynamics have been studied [2], the derivation of hyperjerk form for the classical hyperchaotic systems has not been investigated.

The hyperjerk dynamical systems are emerging as the simplest class of dynamical systems with hyperchaotic behavior. In this paper, we review the hyperjerk form of hyperchaotic dynamics of some classical hyperchaotic circuits showing that for some of them, even if the hyperjerk form can be achieved it is possible to do it only with respect to some variables, while for other circuits the hyperjerk form does not exist with respect to any state variable.

In particular, the Rossler system [3], the Lorenz system [4], the Lü system [5], and the Chen circuit [6] have been analyzed and the derivation of the corresponding possible hyperjerk forms considered. It will shown that hyperjerk dynamics can be obtained in the first three cases, while for the hyperchaotic Chen circuit can not be derived. In fact even if starting from a differential equation of degree $n$, a system with $n$ state variables can always be derived, in general the viceversa is not true.

Furthermore, a fundamental aspect in chaotic and hyperchaotic systems is also addressed, i.e. the synchronization of a pair of hyperjerk systems [7, 8, 9]. Aim of this paper is also to propose a strategy for synchronization which exploits the particular structure of an hyperjerk system.

The paper is organized as follows: in Sec. 2 the hyperjerk form od some hyperchaotic dynamical system are roported. In Sec. 3 the synchronization of hyperjerk has been studied. Moreover in the conclusive remarks some results on deriving hyperjerk forms will be emphasized in order to propose a general synchronization approach.

## 2. Hyperjerk form for hyperchaotic circuits

In this section the main hyperchaotic systems reported in the literature will be studied. In particular, it has been studied the possibility of deriving the hyperjerk form for four canonical hyperchaotic circuits. In principle, considering fourth-order dynamical systems, four different hyperjerk forms can be obtained from each system, consisting in the following fourth-order differential equation:

$$\ddddot{X} = f(\dddot{X}, \ddot{X}, \dot{X}, X) \tag{1}$$

where $X$ is the generic state variable and $f$ is a nonlinear function of the other derivatives of the same state variable.

### 2.1. Rossler hyperchaotic system

The hyperchaotic extension of the well-known Rossler oscillator is described by the following dynamical equations:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay + w \\ \dot{z} &= b + xz \\ \dot{w} &= -cz + dw \end{aligned} \tag{2}$$

where $a = 0.25$, $b = 3$, $c = 0.5$, and $d = 0.05$ are parameter values for which an hyperchaotic behavior can be observed. We first try to derive an hyperjerk form of the type (1) for state variable $x$. Let us start from the first equation of (2), from which we can derive $z = -y - \dot{x}$. The strategy is to operate successive differentiation with respect to time of the first equation of (2) and derive the other two state variable as a function of $x$ and its derivatives. Hence, differentiating the first equation once we obtain $\ddot{x} = -\dot{y} - \dot{z} = -x - ay - w - b + xy + x\dot{x}$ from which we derive $w = -\ddot{x} - x - ay - b + xy + x\dot{x}$, differentiating the same equation twice we derive $y = \frac{\dddot{x} + (1+(a+d)x+c-x^2)\dot{x} - (a+d-x)\ddot{x} + (b-d)x - ab - db}{x^2+\dot{x}-(a+d)x+ad-c}$. Finally, the hyperjerk form can be achieved with a further differentiation obtaining

$$\begin{aligned}\ddddot{x} &= (x+ay+w)(x^2+\dot{x}-(a+d)x+ad-c)\\ &+y(\ddot{x}+2x\dot{x}-(a+d)\dot{x})+(2x\dot{x}-(a+d)\dot{x}+b-d)\dot{x}\\ &-(1+c+(a+d)x-x^2)\ddot{x}-\dot{x}\ddot{x}+(a+d-x)\dddot{x}\end{aligned} \tag{3}$$

in which $y$, $z$, and $w$ can be substituted with the expressions derived above.

Following the same procedure, the hyperjerk forms with respect to all the other three state variables can be found.

### 2.2. Lorenz hyperchaotic system

The hyperchaotic formulation of the Lorenz system is governed by the following set of equations:

$$\begin{aligned}\dot{x} &= a(y-x)+w\\ \dot{y} &= rx-y-xz\\ \dot{z} &= xy-bz\\ \dot{w} &= dw-\beta xz\end{aligned} \tag{4}$$

where $a = 10$, $r = 28$, $b = 8/3$, $d = 1.7$, $\beta = 1$ are parameter values for which hyperchaos arises.

From the first equation of (4 we derive $w = \dot{x} - ay + ax$. Differentiating the first equation and substituting $w$, and $\dot{y}$ and $\dot{w}$ as in Eqs. (4) we have $\ddot{x} = (ar+da)x - a(1+d)y - (a+\beta)xz + (d-a)\dot{x}$, from which $y$ can be derived. Differentiating two times the first equation of (4 and substituting $w$, $y$ and $\dot{y}$, $\dot{w}$ and $\dot{z}$ we obtain $\dddot{x} = (d-a-1)\ddot{x} + (ar+da+d-a)\dot{x} + (da-adr)x - (\beta - ad - b(a+\beta))xz - (a+\beta)\dot{x}z - \frac{a+\beta}{a+ad}x^2(-\ddot{x} + (d-a)\dot{x} + (ar+da)x) + \frac{(a+\beta)^2}{a+ad}x^3 z$. The third state variable can be easily derived as $z = [\dddot{x} + (d-a-1)\ddot{x} + (ar+da+d-a)\dot{x} + (da-adr)x - \frac{a+\beta}{a+ad}x^2(-\ddot{x} + (d-a)\dot{x} + (ar+da)x)]/[(\beta - ad - b(a+\beta))x + (a+\beta)\dot{x} - \frac{(a+\beta)^2}{a+ad}x^3]$. Hence, the hyperjerk form of the Lorenz hyperchaotic system can be written as

$$\begin{aligned}\ddddot{x} &= (d-a-1)\dddot{x} + (ar+da+d-a)\ddot{x}+\\ &+(da-adr)\dot{x} - (\beta - ad - b(a+\beta))\cdot\\ &\cdot[\dot{x}z + x(xy-bz)] - (a+\beta)(\ddot{x}z + \dot{x}(xy-bz))-\\ &-\tfrac{a+\beta}{a+ad}[2x\dot{x}(-\ddot{x} + (ar+da)x + (d-a)\dot{x})+\\ &+x^2(-\dddot{x} + (ar+da)\dot{x} + (d-a)\ddot{x})]+\\ &+\tfrac{(a+\beta)^2}{a+ad}[3x^2\dot{x}z + x^3(xy-bz)]\end{aligned} \tag{5}$$

The same procedure leads to the hyperjerk form for the Lorenz system with respect to $y$. However, it is not possible to derive the hyperjerk form with respect to $z$ and $w$. This is due to the fact that neither from the third equation nor from the fourth, a linear expression for the other state variables can not be derived.

### 2.3. Lü hyperchaotic system

The Lü system is governed by the following dynamical equations:

$$\begin{aligned}\dot{x} &= a(y-x)+w\\ \dot{y} &= -xz+cy\\ \dot{z} &= xy-bz\\ \dot{w} &= xz+dw\end{aligned} \tag{6}$$

where $a = 36$, $b = 3$, $c = 20$, and $d = 1.3$ are the parameter values corresponding to an hyperchaotic attractor. The procedure to derive the hyperjerk forms is the same as described before and can be applied to the derivation of the two hyperjerk forms with respect to $x$ and $y$. The first hyperjerk form is here reported starting from the first equation, from which we derive $w = \dot{x} - ay + ax$. Differentiating once we have $\ddot{x} = (-ax + ac - ad)y - (a-d)\dot{x} + dx + xz$ from which $y = \frac{\ddot{x} + (a-d)\dot{x} - dx - xz}{ac - ad - ax}$. Differentiating twice we get $\dddot{x} = y(-a\dot{x} + ax^2 - acx - (ac-ad)\dot{x} + ac^2 - acd + x^2) - (a-d)\ddot{x} + d\dot{x} - bxz + x\dot{z}$ from which $z$ can be derived, and the hyperjerk form can be found with a further derivation of the first of Eqs. (6).

### 2.4. Chen hyperchaotic system

Finally, let us consider the case of the hyperchaotic Chen system, which represent a slight modification of the Lorenz system. In particular, the three nonlinearities of the system are three different cross-product between state variable:

$$\begin{aligned}\dot{x} &= a(y-x)+w\\ \dot{y} &= dx-xz+cy\\ \dot{z} &= xy-bz\\ \dot{w} &= yz+rw\end{aligned} \tag{7}$$

where $a = 35$, $b = 3$, $c = 12$, $d = 7$, and $r = 0.1$. The presence of three different nonlinearities prevent the possibility of reaching any hyperjerk.

## 3. Hyperjerk synchronization remarks

Even if classical schemes of synchronization can be used for hyperjerk systems, the study of observed-based synchronization here will be investigated. Moreover, in order to evaluate a simple scheme the possibility of synchronizing two of them by stabilizing the linear part of the hyperjerk form will be studied. For the hyperjerk systems introduced in the previous section a linear observer can not achieve the synchronization while for the hyperjerk system it is possible.

Let us consider, a general hyperjerk form with respect to a given state variable and rewrite it as a set of four first-order nonlinear differential equations:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= a_0x_1 + a_1x_2 + a_2x_3 + a_3x_4 + g(x_1,x_2,x_3,x_4)\end{aligned} \tag{8}$$

where $g$ is a nonlinear function. The state matrix $A$ of the linear part of system in Eqs. (11) can be easily derived as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}$$

A strategy based on the definition of an observer for the linear part can be easily implemented acting on the eigenvalues of matrix $A$. Stabilizing matrix $A$, the observer can be written as:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1 e \\ \dot{\hat{x}}_2 &= \hat{x}_3 + k_2 e \\ \dot{\hat{x}}_3 &= \hat{x}_4 + k_3 e \\ \dot{\hat{x}}_4 &= a_0\hat{x}_1 + a_1\hat{x}_2 + a_2\hat{x}_3 + a_3\hat{x}_4 + g(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) + k_4 e \end{aligned} \quad (9)$$

where $k_1$, $k_2$, $k_3$, and $k_4$ are observer gains and $e = x_1 + x_2 + x_3 + x_4 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 - \hat{x}_4$ is the fed-back error.

However the onset of a synchronous behavior can be achieved only in particular cases. If we consider the hyperjerk form reported for the Lorenz hyperchaotic system, in fact, the state matrix of the linear part can be written as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 459 & -288.7 & 9.3 \end{bmatrix}$$

but stabilizing it with a gain vector $\mathbf{k} = \begin{bmatrix} -0.1344 & 46.8852 & -26.9273 & -0.5236 \end{bmatrix}$ the observer does not converge to a stable synchronous solution.

It is worth noticing that considering an hyperchaotic system originally written in hyperjerk form, namely the hyperchaotic snap system [1], the strategy based on the observer reveals its effectiveness. In fact, given the following hyperjerk form:

$$\ddot{\ddot{x}} + x^4\dddot{x} + a\ddot{x} + \dot{x} + x = 0 \quad (10)$$

where $a = 3.6$ is the single bifurcation parameter, can be rewritten as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_1 - x_2 - ax_3 - x_1^4 x_4 \end{aligned} \quad (11)$$

The state matrix of the linear part can be stabilized using a gain vector $\mathbf{k} = \begin{bmatrix} 5.0000 & 5.4545 & 2.7879 & -3.2424 \end{bmatrix}$ and a stable synchronous behavior can be observed as shown in Figs. 1 and 2.

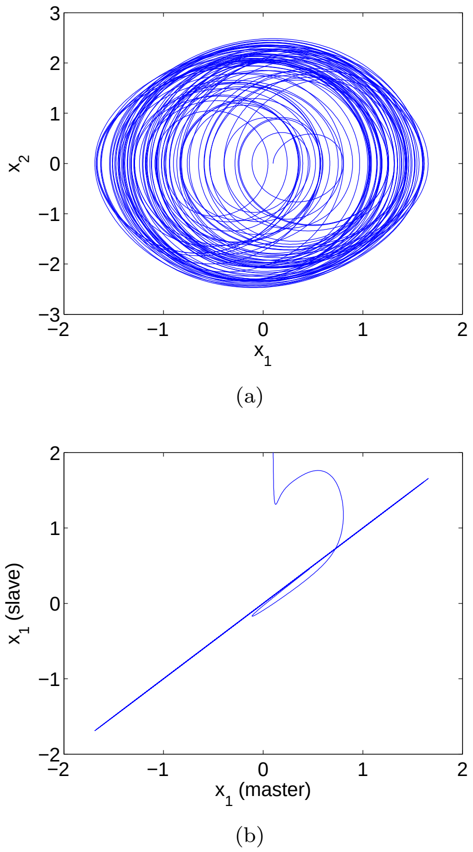


Figure 1: Synchronization of two snap hyperchaotic systems using the observer-based approach: (a) hyperchaotic attractor and (b) synchronization plot.

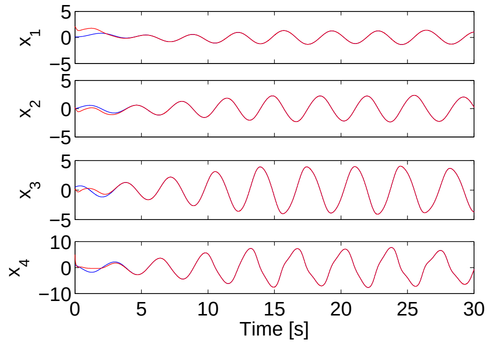


Figure 2: Synchronization of two snap hyperchaotic systems using the observer-based approach: temporal evolution of the four state variables.

## 4. Conclusion

In this paper hyperjerk forms of classical hyperchaotic systems are introduced. Moreover a study on the synchronization of hyperjerk forms has been proposed showing that exploiting the particular characteristic of the jerk form allows to define a strategy based on linear observer. It is shown that numerical evidences allow to postulate that this strategy, based on the stabilization of the linear part of the jerk form, is suitable only for hyperchaotic systems originally expressed in hyperjerk form.

## References


[1] K. E. Chlouverakis, J. C. Sprott "Chaotic hyperjerk systems," *Chaos, Solitons and Fractals*, vol.28, pp.739–746, 2006.

[2] J. C. Sprott *Elegant Chaos*, World Scientific 2010.

[3] O. E. Rossler "An equation for hyperchaos," *Physics Letters A*, vol.71, pp.155–157, 1979.

[4] R. Barboza "Dynamics of a hyperchaotic Lorenz system," *Int. J Bif and Chaos*, vol.17, pp.4285–4294, 2007.

[5] J. Lü, G. Chen "A new chaotic attractor coined," *Int. J Bif and Chaos*, vol.12, pp.659–661, 2002.

[6] Y. Li, S. W. Tang, G. Chen "Generating hyperchaos via state feedback control," *Int. J Bif and Chaos*, vol.15, pp.3367–3375, 2005.

[7] L. M. Pecora, T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, pp. 821–824, 1990.

[8] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, C. S. Zhou, "The synchronization of chaotic systems," *Phys. Rep.*, vol. 366, pp. 1–101, 2002.

[9] A. Buscarino, L. Fortuna, M. Frasca, "Experimental robust synchronization of hyperchaotic circuits," *Physica D*, vol. 238, pp. 1917–1922, 2009.