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Bifurcation Analysis of Coupled Nagumo-Sato Models

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Abstract—The Nagumo-Sato model is one of mathematical neuron models described by a piecewise linear difference equation. Since there is a conditional character which is discontinuous at the threshold value, the system can be classified as a hybrid dynamical system. Bifurcation phenomena are occurred by changing internal parameters and chaotic attractors are also given. The dynamical properties were exactly studied analytically.

In this paper, we investigate the bifurcations of diffusively-coupled Nagumo-Sato models. By using complementarity a shooting algorithm and brute-force method, complete bifurcation diagrams are obtained. In spite of the discontinuities inside the coupled system, our shooting method can solve bifurcation problems. A period-locking regions edged by border-collision bifurcation sets are found, and chaotic regions are distinguished by a tangent bifurcation. We discuss on changing bifurcation structures with parameter variations.

1. Introduction

A Nagumo-Sato model is one of mathematical neuron models[1, 2] written as follows:

$$x_{k+1} = f(x_k) \tag{1}$$

$$f(x) = \begin{cases} ax - b + 1 & (x < C) \\ ax - b & (x \geq C) \end{cases} \tag{2}$$

This model is included in discrete-time piecewise affine systems. In these systems, a flow as a solution of a difference equation is suddenly switched to another difference equation by getting across the system border. In control engineering field, if a state space has some non-smooth characteristics, the system is called a hybrid system. Here, discrete-time piecewise affine systems is categorized in hybrid systems. Thus the Nagumo-Sato model is also regarded as one of hybrid systems[3].

Bifurcation problems on limit cycles observed in a hybrid system is computable if the Poincaré section is defined on the manifold given by the condition of non-smoothness characteristics and a suitable transformation (projection) of the state into local coordinate system[4] is provided. However, as far as authors know, less discussion has been done on bifurcations in hybrid discrete systems.

Various coupled systems has been researched in neuron models. Especially, the gap junction is an important things of the structure of the neuron.

In this paper, we analyze bifurcations of a coupled system of Nagumo-Sato models in the sense of a natural extension for continuous-time gap junction systems as follows:

$$\begin{cases} x_{k+1} = f(x_k) + k(x_k + y_k) \\ y_{k+1} = f(y_k) + k(y_k + x_k) \end{cases} \tag{3}$$

We compute bifurcation parameter values by the shooting method, and a lot of border-collision bifurcations are found[5]. We compare bifurcations of the single Nagumo-Sato model with coupled Nagumo-Sato model, and discuss the characteristics of the model.

2. Bifurcations

2.1. Variational equations

Even the characteristics of the system contains non-smoothness such as hysteresis, break points, we can compute bifurcation parameter value numerically by using a shooting method unless the derivative of the characteristics is not defined, i.e., a smoothness of the characteristics for the state is not required. Rewrite the system (3) as

$$\boldsymbol{x}_j(k+1) = \boldsymbol{f}_j(\boldsymbol{x}(k)), \tag{4}$$

where $j = 1, 2, \ldots, m$, k is a discrete time, $\boldsymbol{x}(k) = (x(k), y(k))$, and each $\boldsymbol{f}_j : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is smooth. There are borders given by $q_j(\boldsymbol{x}) = 0$ and it switches the system from $\boldsymbol{f}_j$ to $\boldsymbol{f}_{j+1}$ if the state $\boldsymbol{x}(k)$ is given by exceeding any border from $\boldsymbol{x}(k-1)$.

Let us denote the solution of Eq.(4) as $\boldsymbol{x}(k) = \boldsymbol{\varphi}(\boldsymbol{x}_0, k)$. It satisfies the initial condition $\boldsymbol{x}(0) = \boldsymbol{\varphi}(\boldsymbol{x}_0, 0) = \boldsymbol{x}_0$. The non-smoothness of Eq. (4) affects the derivative of the solution. In fact, a global derivatives cannot be obtained, however, according to the position of $\boldsymbol{x}$, we can split the system (4) into two linear systems. Similarly, the derivatives of period-n solutions, i.e., the solutions of the first variational equations can be evaluated by solving the following:

$$\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}(k+1) = \frac{\partial \boldsymbol{f}^n}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}(k), \quad \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0}(0) = \boldsymbol{I} \tag{5}$$

where $\partial \boldsymbol{f}^n/\boldsymbol{x}$ is a Jacobian matrix, but a careful evaluation is required to compute it since it depends on locations of periodic points thus it is not a fix matrix.

2.2. Border-collision bifurcations

Border-collision bifurcations are occurred regardless of local characteristics of fixed/periodic points. So border-collision bifurcations are not obtained by the method based on eigenvalues. When the an orbit of the attractor hits with the border in the system, the border-collision bifurcation is occurred. The conditions are written as:

$$\begin{cases} \boldsymbol{\varphi}(\boldsymbol{x}_0, \lambda) - \boldsymbol{x}_0 = 0 \\ f(x_0, \lambda) - C = 0 \end{cases} \quad (6)$$

The point of border-collision bifurcation x_0 and the parameter value λ can be obtained by solving Eq. (6).

2.3. Bifurcation diagrams

Figure 1 shows one dimensional bifurcation diagram and the maximum Lyapunov exponent of Eq. (3), when parameters are $b = 0.5$, $C = 0.5$ and $k = 0.1$, the initial value is $(x, y) = (0.6, 0.1)$. In this figure, a is increased from $a = 0$, the orbit of x changes from period-2 to period-3, when it is occurred a bifurcation. However, Lyapunov exponent is not 0 when this bifurcation occurs, therefore it has a possibility of the border-collision bifurcation. In Fig. 1, zero Lyapunov exponent is happened when $x = 0.8$, and further increment of a there, chaos is observed. It is related with the tangent bifurcation.

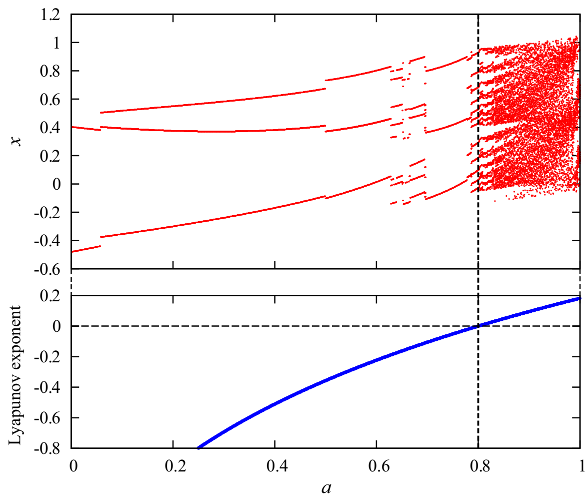

Figure 1: One dimensional bifurcation diagram (red) and Lyapunov exponent (blue)

Figure 2 shows example of the situation of border-collision bifurcation in the vicinity of $a = 0.61$ in Fig. 1. Red points is a period-3 attractor and green points is a period-8 attractor. If the period-3 attractor moves along the arrow by a parameter variation, by hitting the border $x = C$, we have suddenly the period-8 attractor. This border-collision bifurcation does not have a bistable situation.

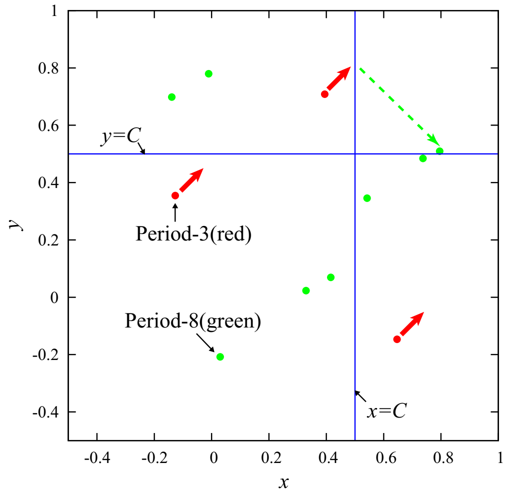

Figure 2: Situation of border-collision bifurcation

Figure 3 is the bifurcation diagram of coupled Nagumo-Sato model. Symbols mean:

- T : Tangent bifurcation.
- B_{nk} : Border-collision bifurcation of period-n, k is the reference number.

We calculate several bifurcation curves Fig. 3. We observe that a lot of border-collision bifurcations are occurred from the Eq. (3) and they are computable solving Eq. (6). There is an big island of period-1 on the upper left, and period-2 on lower left. When a increases, it is occurred border-collision bifurcations and consists of high-periodic areas. It is noteworthy that border-collision bifurcations B_{11}, B_{31}, B_{32}, B_{33} and B_{34} do not terminate on the tangent bifurcation T, and they lie on chaotic area. Thus they are regarded as bifurcations for unstable periodic points.

3. Characteristics

We compare the bifurcations of single Nagumo-Sato model with coupled it. Figure 4 is bifurcation diagram of Nagumo-Sato models, there are (a)single and (b)coupled, and Fig. 5 is enlarged diagram of Fig. 4 that the vicinity of tangent bifurcation. In Fig. 4, we use brute-force method to compute bifurcation diagrams, because hi-periodic fields have been overcrowded. In this figure, the horizontal and vertical axes are a and b respectively, where $0.0 < a < 1.0$, $0.0 < b < 1.0$. The blue region indicates the parameter region which shows a period-1 (fixed point) in the state space. In the same way, the red region is period-2, the green is period-4, and the black region shows a chaotic area, if over

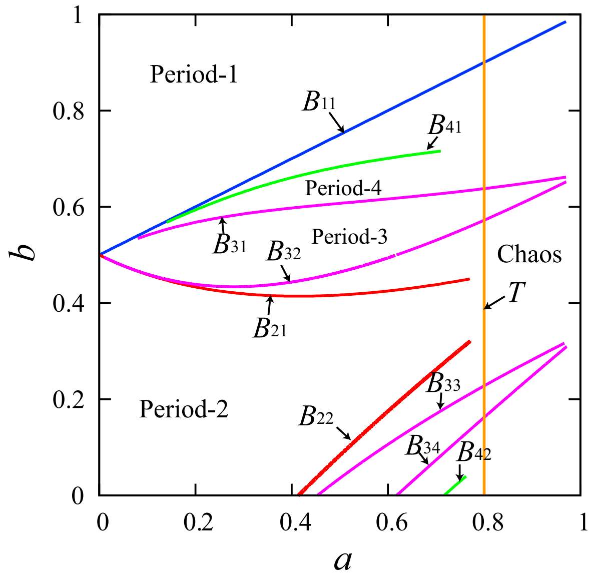

Figure 3: Bifurcation diagram of the coupled Nagumo-Sato model

period-13 when coloring by color of the remainder divided by 13.

In Figure 4 and 5, as for the coupled model, the total number of border-collision bifurcations decreases compared with the single model, because a lot of border-collision bifurcations are occurred by parameter $a \approx 1.0$, however the tangent bifurcation is occurred with $a = 0.8$. Where the Jacobian matrix of Eq. (3) are:

$$\frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{x}_1} = \begin{pmatrix} a+k & -k \\ -k & a+k \end{pmatrix}, \quad \frac{\partial \boldsymbol{f}_2}{\partial \boldsymbol{x}_2} = \begin{pmatrix} a+k & -k \\ -k & a+k \end{pmatrix}, \tag{7}$$

and the eigenvalue of Eq. (7) is:

$$\mu = a + k \pm \sqrt{(a+k)^2 - a^2 + 2ak}. \tag{8}$$

The position of tangent bifurcation is decided depending on the coupling factor k, because is decided by the eigenvalue μ. Figure 6 is the bifurcation diagram that parameter $k = 0.2$ and $k = 0.4$, and Fig. 7 is the bifurcation diagram that the horizontal and vertical axes are a and k. The tangent bifurcation where is border of the chaotic area is changed as coupling factor k.

Figure 8 is the phase portrait of the chaotic area. In Fig. 8 (a) when parameter $a = 0.9$, the chaos appears to both sides across $y = x$, however (b) when $a = 1.1$, the chaos appears that is wide range and unsteady.

4. Conclusions

We compute bifurcation sets of the coupled Nagumo-Sato model by the shooting algorithm as the hybrid system, and we show a lot of border-collision bifurcations and the tangent bifurcation. It is clarified that the size of the chaotic region is radically depended on the value of parameter a.

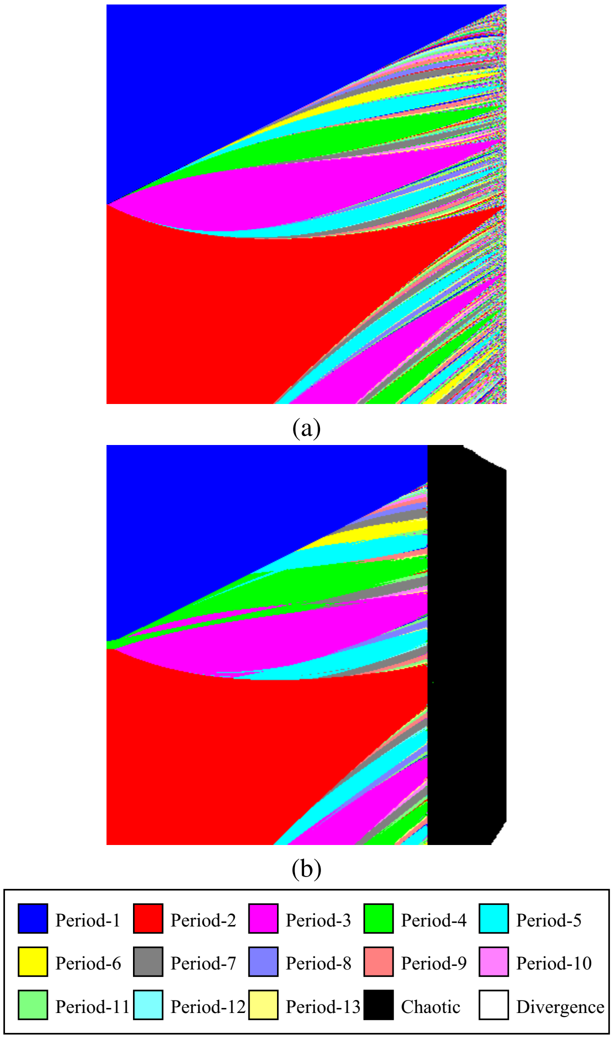

Figure 4: (a) Bifurcation diagram of the single Nagumo-Sato model, parameters are $0.0 < a < 1.0$, $0.0 < b < 1.0$, $C = 0.5$. (b) Bifurcation diagram of the coupled Nagumo-Sato model, in the parameter range with $k = 0.1$

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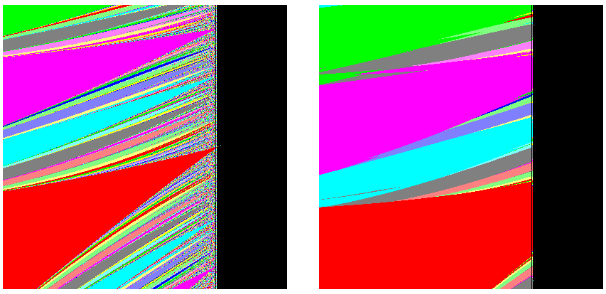
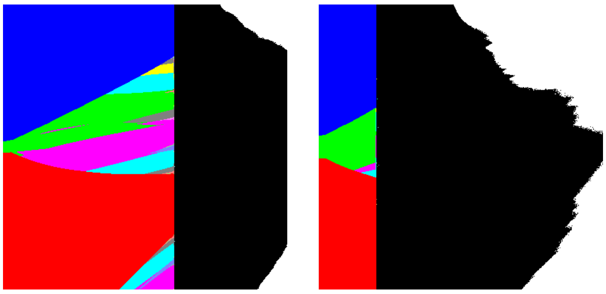


(a) (b)

Figure 5: Enlarged diagram of bifurcation diagram, the vicinity of tangent bifurcation. (a) single model, $0.7 < a < 1.1$, $0.3 < b < 0.7$, $C = 0.5$. (b) coupled model, $0.5 < a < 0.9$, $0.3 < b < 0.7$, $C = 0.5$, $k = 0.1$.



(a) (b)

Figure 6: Bifurcation diagrams (a) $0.0 < a < 1.0$, $0.0 < b < 1.0$, $k = 0.2$. (b) $0.0 < a < 1.0$, $0.0 < b < 1.0$, $k = 0.4$.

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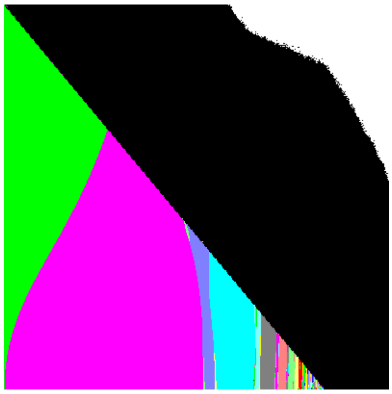
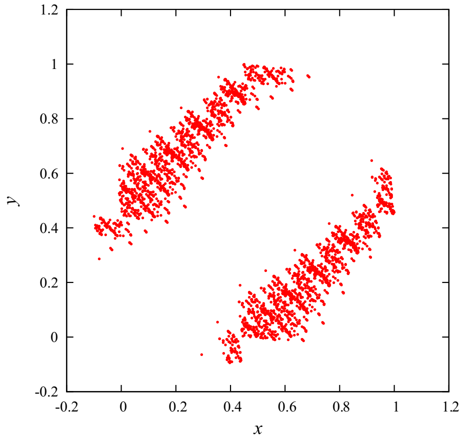
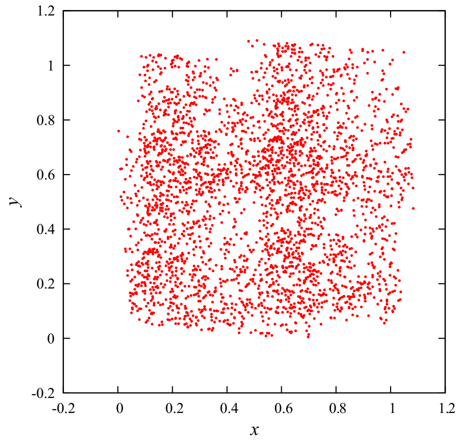


Figure 7: Bifurcation diagram, $0.0 < a < 1.0$, $0.0 < k < 0.5$, $b = 0.5$ $C = 0.5$.



(a)



(b)

Figure 8: Phase portraits of chaos fields (a) parameters are $a = 0.9$, $b = 0.5$, $C = 0.5$, $k = 0.1$ (b) $a = 1.1$ other is the same