Compressive Sensing with Measurement Matrix Constructed by Cat Map

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Abstract – A new approach to construct the measurement matrix in compressive sensing is proposed. In this approach, a circulant matrix is first generated by a modified cat map and a random sequence. Then it is optimized through solving the problem of sparse optimization and becomes the measurement matrix used in compressive sensing. The proposed measurement matrix is compared with the Bernoulli and chaos-based random measurement matrices before and after optimization. The results show that our construction method leads to a higher reconstruction fidelity.

1. Introduction

According to the Nyquist-Shannon theorem[1], a signal cannot be exactly recovered from its discrete samples unless the sampling frequency is at least two times the highest signal frequency. However, this is not necessary if some high frequency components are ignored after the sampled signal has been compressed.

Recently, a new sampling theory called compressive sensing (CS) was proposed by Candes *et al.*[2-5] and Donoho[8] independently. Its principle is that a smaller measurement matrix can be used to sample a sparse signal in order to extract the significant information of this sparse signal only. The exact reconstruction of the sparse signal is guaranteed if some conditions are satisfied. In this way, the measurement matrix is far smaller than that required by the Nyquist-Shannon theorem.

The core of CS is that sampling and compression can be performed at the same time. The measurement matrix used in the sampling process needs to be incoherent with the sparse transform basis of the signal. More formally, the Restricted Isometry Property (RIP) must be satisfied [6]. It is well-known that a random matrix is incoherent with almost any matrices and it satisfies the RIP criterion with a high probability.

According to the above criteria, various methods for constructing the measurement matrix have been suggested. They can be divided into three categories. The first one includes Bernoulli random measurement matrix[7-9], very sparse random projection matrix[10], etc. The elements in these matrices are independent but follow a certain distribution. Their size is very small but the computational complexity is high. The second category consists of the partial Fourier matrix[5,11], non-relevant measurement matrix[12], etc. These matrices are generated through extracting the rows of an orthogonal matrix. These algorithms are faster than those belonging to the first category. However, they are incoherent with most sparse signals. The final category includes binary sparse matrix[13], structurally random matrix[12], chirp measurement matrix[14], etc. This kind of matrices has a certain kind of deterministic structure. The time required for generating the measurement matrix and reconstructing the original signal is shorter than that of the other two kinds. However, the quality of the recovered signal is not satisfactory.

In this paper, an approach for constructing a measurement matrix having the advantages of those in the first and the third categories is proposed. In particular, a circulant matrix satisfying RIP is constructed using a uniformly-distributed sequence generated by a modified cat map. Then, the matrix is optimized to enhance the reconstruction fidelity through solving the sparse optimization problem.

The main contribution of this work is the establishment of a connection between CS and the chaotic cat map. The performance of the suggested approach with and without optimization is compared with that of Bernoulli and chaotic measurement matrices on the reconstruction fidelity. The experimental results justify that the proposed measurement matrix leads to a better reconstruction than the other two matrices. This paper is organized as follows. In Section II, the proposed method for constructing the measurement matrix is introduced. The approach to optimize the matrix generated by the modified cat map and the property of the resultant matrix are analyzed in Section III. Simulation results are presented in Section IV. The final section concludes our work.

2. Construction and Optimization of Measurement Matrix

2.1. Matrix Construction

The classic cat map was proposed by Arnold and Avez[15]. It has been widely used in image encryption as it can re-distribute all pixels of the original image to make the image noise-like and unrecognizable. The cat map is governed by the following equation:

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & ab+1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \bmod n \qquad (1)$$

where (x_i, y_i) is the coordinate of a pixel in the original image, (x_{i+1}, y_{i+1}) is the coordinate of the corresponding pixel in the encrypted image. The image size is $n \times n$ while a and b are the cat map parameters. The determinant of the matrix $\begin{bmatrix} 1 & a \\ b & ab+1 \end{bmatrix}$ is 1 to guarantee the property of one-to-one mapping.

Equation (1) suggests that the image is a square one. However, the measurement matrix used in CS is not a square matrix. Therefore, the cat map should be modified in the following way to construct the circulant matrix.

$$\begin{bmatrix} x'_{i+1} \\ y'_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x'_i \\ y'_i \end{bmatrix} \mathrm{mod} \begin{bmatrix} m \\ n \end{bmatrix} \quad (2)$$

where (x'_i, y'_i) is the coordinate of the original matrix element, (x'_{i+1}, y'_{i+1}) is its new coordinate. The size of the measurement matrix is $m \times n$. Equation (2) can be expressed as

$$\begin{cases} x'_{i+1} = x'_i \bmod m \\ y'_{i+1} = (x'_i + y'_i) \bmod n \end{cases} \quad (3)$$

To construct the measurement matrix $\Phi \in R^{m\times n}$, a source with uniform distribution is chosen to generate a random sequence $Z = \{z_1, z_2, \cdots, z_n\}$. Then the matrix Δ is constructed row-by-row using this sequence.

$$\Delta = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_n \\ z_1 & z_2 & z_3 & \cdots & z_n \\ z_1 & z_2 & z_3 & \cdots & z_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & \cdots & z_n \end{pmatrix}_{mn} \quad (4)$$

After that, the matrix Δ is permuted by the modified cat map according to Eq. (3) Then the permuted matrix Δ' is obtained and the measurement matrix Φ is constructed by the following equation

$$\Phi = \frac{1}{p} \cdot \Delta' = \frac{1}{p} \cdot \begin{pmatrix} z_n & z_1 & z_2 & \cdots & z_{n-1} \\ z_{n-1} & z_n & z_1 & \cdots & z_{n-2} \\ z_{n-2} & z_{n-1} & z_n & \cdots & z_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{n-m+1} & z_{n-m+2} & z_{n-m+3} & \cdots & z_{n-m} \end{pmatrix}_{mn} \quad (5)$$

where p is a parameter.

2.2. Matrix Optimization

In CS, the coherence between the measurement matrix Φ and the transform basis Ψ determines the sparse boundary of the given signal, as given by

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu_{mx}}\right) \quad (6)$$

where x is a sparse matrix and μ_{mx} is the coherence.

In effective compressive sampling, low coherence leads to high quality of the reconstructed signal. Therefore, we reduce the coherence between the measurement matrix and the transform basis through solving the sparse optimization problem [16]. The coherence is the value of the largest elements in the Gram matrix constructed by the measurement matrix and the transform basis, as defined by

$$\mu_{mx} = \max_{i\neq j, 1\leq i, j \leq n} |g_{ij}| \quad (7)$$

where g_{ij} is the element on the i^{th} row and the j^{th} column of the Gram matrix $G = \Psi^T \Phi^T \Psi \Phi$.

When the coherence[17-22] is the smallest, it follows from Eq. (7) that the ideal Gram matrix is an identity matrix, i.e.,

$$G = I = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \quad (8)$$

Before the measurement matrix is optimized, the Gram matrix will be initialized. The gap between the initial Gram matrix and the identity matrix can be narrowed by iterations through the unitary constant paradigm. Thus, the optimization of the initial measurement matrix can be realized.

However, due to the intrinsic characteristics of the identity matrix, it is very difficult to make the initial Gram matrix constantly approximate the identity matrix. It is necessary to relax the criteria in order to construct a more reasonable ideal matrix, such as the following one:

$$G' = \begin{pmatrix} 1 & \cdots & g'_n \\ \vdots & \ddots & \vdots \\ g'_n & \cdots & 1 \end{pmatrix} \quad (9)$$

where G' is an $n \times n$ symmetric matrix and μ is the threshold value such that $\max_{i\neq j} |g'_{ij}| \leq \mu$.

The procedures for optimizing the measurement matrix are described as follows. Firstly, the measurement matrix and the transform basis are utilized to construct the Gram matrix:

$$D = \Phi\Psi \quad (10)$$

$$G = D^2 = D^T D \quad . \quad (11)$$

Secondly, an identity matrix is initialized as a Gram matrix G', which is further updated according to Eq. (12).

$$\forall i, j, i \neq j : g'_{ij} = \begin{cases} g_{ij}, & if\ |g_{ij}| \leq \mu \\ \mu \cdot sign(g_{ij}), otherwise \end{cases} \quad (12)$$

Finally, optimize and update the measurement matrix Φ using the criterion stated in Eq. (13).

$$\min \|G - G'\|_F^2 \quad (13)$$

Then we can obtain

$$\Phi_{(k+1)} = \Phi_{(k)} - \beta \Phi_{(k)} \Psi (\Psi^T \Phi^T{}_{(k)} \Phi_{(k)} \Psi - G') \Psi^T \quad (14)$$

where β is the iteration step size.

Through the optimization method, the coherence between the measurement matrix and the transform basis can be effectively reduced.

3. Properties of the Measurement Matrix

It is widely known that there are two criteria in constructing the measurement matrix: RIP and the incoherence between the measurement matrix and the transform basis. The original signal x can be reconstructed using the sparse decomposition algorithm when the measurement matrix Φ and the transform basis Ψ satisfy RIP, as given by

$$(1-\delta_k)\|A\|_2^2 \le \|\Phi A\|_2^2 \le (1+\delta_k)\|A\|_2^2 \tag{15}$$

where δ_k is an iso-volumetric constant with range $\delta_k \in (0,1)$ and $A = \Psi x$.

However, RIP is difficult to prove and so the incoherence is put forward. The sparse boundary of the signal under sampling is determined by the coherence. The more the non-zero coefficients the sparse matrix Ψx contains, the more significant the information content that the measured signal y possesses. The fidelity of the reconstructed signal is also higher. According to Eq. (6), the bigger the sparse boundary of the original signal x, the less the coherence is.

The range of the coherence μ is given by:

$$\mu \in [0,1] \tag{16}$$

The measurement matrix constructed by the proposed method satisfies the two criteria, as determined by μ in the optimization process.

4. Simulation Results

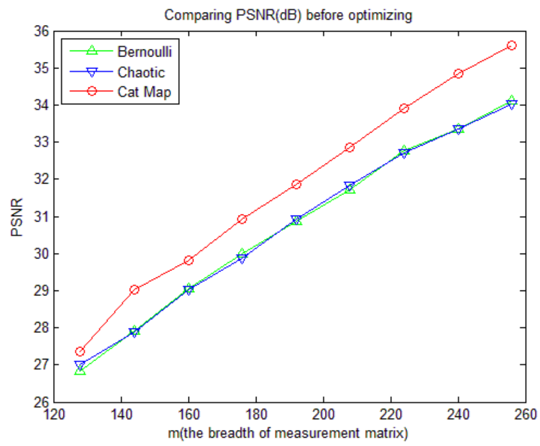
The proposed method for constructing the measurement matrix with the modified cat map and a uniformly-distributed source is realized using Matlab. In the experiment, the 256×256 Lena image is selected as the test image while Discrete Wavelet Transform (DWT) is chosen as the sparse transformation. The uniformly-distributed source is employed to generate the random sequence which is used in the construction of the measurement matrix. The parameter settings are p=10, μ=0.2 and β=0.01. Optimization of the measurement matrix is achieved by the iterative computation. The signal reconstruction algorithm is Optimal Matching Pursuit (OMP) which ensures a better reconstruction performance at a smaller number of iterations.

Moreover, Bernoulli and chaotic measurement matrices are constructed to compare with the proposed measurement matrix. According to [2, 23], the general construction methods for these two measurement matrices are, respectively,

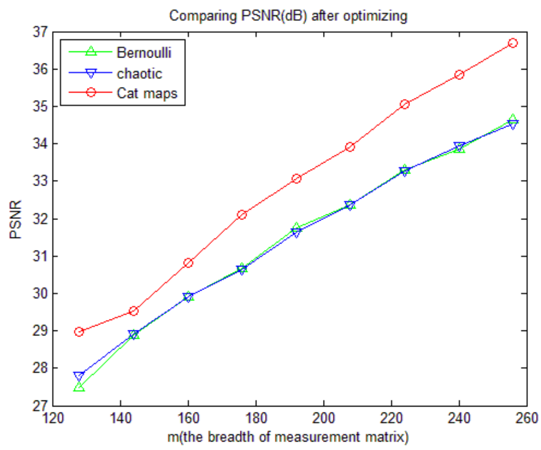
Bernoulli random measurement matrix: $B \in R^{M\times N} : B(i,j) = \frac{1}{\sqrt{m}} b_{ij}, b_{ij} \sim \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$;

Chaotic measurement matrix: use the logistic sequence to produce $Z(d,k,z_0) = \{z_n, z_{n+d}, \cdots, z_{n+kd}\}$ and then construct the measurement matrix $\Phi = \sqrt{\frac{2}{m}} \begin{pmatrix} x_0 & \cdots & x_{m(n-1)} \\ \vdots & \ddots & \vdots \\ x_{m-1} & \cdots & x_{mn-1} \end{pmatrix}, x_k = 1 - 2z_{n+kd}$.

Ten experiments for each of the three kinds of measurement matrix at different sampling rates have been conducted. The average Peak Signal-to-Noise Ratio (PSNR) value is employed to evaluate the quality of the reconstructed image. The results before and after optimization are plotted in Fig. 1(a) and (b), respectively.



(a)



(b)

Fig. 1 Reconstruction PSNR (dB) for the three types of measurement matrices (a) before, and (b) after optimization.

As observed from Fig. 1, the measurement matrix constructed by the modified cat map results in a higher PSNR at all sampling rates with and without optimization.

When the width *m* of the measurement matrix is 180, the average time for generating the Bernoulli and chaotic measurement matrices without optimization are respectively 8.57s and 7.90s. It takes only 7.29s using our approach which is shorter than that required by the other two comparative schemes. These results justify the superiority of the proposed method.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grants Nos. 61374178 and 61202085), the Liaoning Provincial Natural Science Foundation of China (Grant No. 201202076), the Fundamental Research Funds for the Central Universities (Grant Nos. N130417004 and N130317001).

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