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Expansion of a Software for Analysing Bifurcation Observed in a Non-autonomous System with Square-wave Pulses

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Abstract—"BunKi" is a free software program for analyzing bifurcation phenomena observed in nonlinear dynamical systems. By using the software, we can conduct real-time simulation of nonlinear dynamical systems and can evaluate its stabilities of solutions. In addition, this software enables to track numerically bifurcation sets on an arbitrary two parameter plane. The current version of Bunki supports bifurcation analysis for non-autonomous systems with an external continuous inputs, e.g., sinusoidal input. However, the bifurcation analysis in a non-autonomous system with an external discontinuous input such as a square-wave pulse trains cannot be conducted. Accordingly, we added a new function that allows us to analyze bifurcation phenomena observed in a non-autonomous system with square-wave pulse trains. In this report, we demonstrate to analyze bifurcation of periodic solution observed in the non-autonomous system with square-wave pulse trains, and briefly show how to use BunKi.

1. Introduction

Bifurcation analyses in nonlinear dynamical systems are one of the most powerful analysis methods, to find changes in the stability of periodic oscillations and the emergence of complex oscillations for choosing parameters. BunKi [1] has been developed as a free software for the bifurcation analyses. The software has intuitively easy-to-understand interfaces and can be conducted comparatively easy bifurcation analyses in continuous and/or discrete autonomous systems belonging to nonlinear dynamical systems. To investigate bifurcation phenomena observed in a non-autonomous system with an external continuous input, a numerical and computational method proposed by Kawakami [2] was implemented into the software. However, the method implemented into the software cannot apply to bifurcation analyses of a non-autonomous system when there is a certain kind of discontinuity in the external forcing. For example, a mathematical model to consider the entrainment of the circadian rhythm in the biological system to diurnal change was modeled as the non-autonomous discontinuous system whose the periodic forcing discontinuously changed over time.

Therefore, we added a new function to the software. In particular, we improved the software that enables to analyze bifurcation for a non-autonomous system with square-wave pulse trains. In the following, we apply the new function of the software to a circadian clock model with light-dark cycles as an example of the non-autonomous discontinuous system with square-wave pulse trains, and briefly show an example of the bifurcation analysis of the circadian system by using BunKi.

2. Method for Bifurcation Analysis in a Non-autonomous System with Square-wave Pulse Trains

Bifurcation occurs when the stability of periodic solutions changes by varying system parameters. To investigate these bifurcations, we use a method involving a stroboscopic map, also called the Poincaré map. Thereby, the analysis of a periodic solution is reduced to that of a fixed point on the Poincaré map. In the following, we briefly explain the method of calculating a bifurcation set on an arbitrary two-parameter plane in a non-autonomous system with the square-wave pulse trains.

Now let us consider the following non-autonomous differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, \lambda) \tag{1}$$

where $t \in R$ denotes the time, x is the state $x \in R^n$, and $\lambda \in R^m$ is the system parameters. Let f in Eq. (1) be periodic in time with τ, i.e. $f(t+\tau, x, \lambda) = f(t, x, \lambda)$, for all t. We also assume that a solution of Eq. (1) with an initial condition $x = x_0$ at $t = t_0$ is described by $x(t) = \varphi(t, \lambda; t_0, x_0)$, for all t.

Figure 1(a) shows an example of the square-wave pulse trains, where $s_{\min}$ and $s_{\max}$ represent the minimum and maximum value of the square-wave pulse, respectively and α denotes the ratio of the time remained at the value of $s_{\max}$ to that at the value of $s_{\min}$.

Now, let us redefine the system of Eq. (1) as the non-autonomous system with the square-wave pulse trains as follows:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(t, x, \lambda) \\ &= \begin{cases} f_a(x, \lambda_0, \lambda_a), & t_0 \le t < t_0 + \alpha\tau \\ f_b(x, \lambda_0, \lambda_b), & t_0 + \alpha\tau \le t < t_0 + \tau \end{cases}\end{aligned} \tag{2}$$

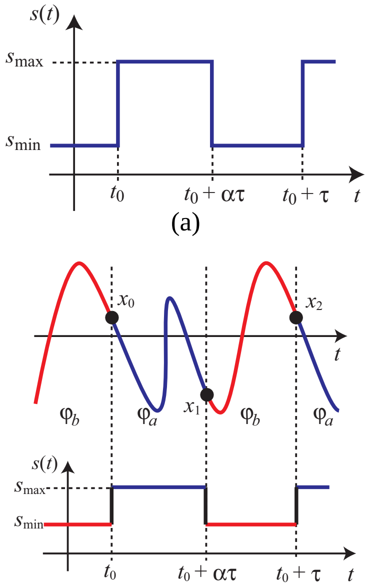


Figure 1: A schematic diagram of square-wave pulse trains and of a discontinuous trajectory observed in a non-autonomous system with periodic discontinuous pulse trains. (a) square-wave pulse trains. (b) Discontinuous trajectory of a periodic solution.

where $\lambda_0 \in R^{m-1}$ denotes common parameters for f, and $\lambda_a, \lambda_b \in R$ are the parameters specifying f_a and f_b, respectively. Namely, the parameters λ_a and λ_b correspond to $s_{\max}$ and $s_{\min}$ of Fig. 1(a), respectively. In the following, we describe the solutions of the first and second equation of Eq. (2):

$$x(t) = \varphi_a(t, \lambda_0, \lambda_a; t_0, x_0), \quad t_0 \le t < t_0 + \alpha\tau, \tag{3}$$

$$x(t) = \varphi_b(t, \lambda_0, \lambda_b; t_0, x_0), \quad t_0 + \alpha\tau \le t < t_0 + \tau. \tag{4}$$

Figure 1(b) shows the schematic diagram of a trajectory consisting of the solutions of Eqs. (3) and (4). The Poincaré map

$$\begin{array}{rcl} S : R^n & \rightarrow & R^n \\ x_0 & \mapsto & S(x_0) = \varphi(t_0 + \tau, \lambda_0, \lambda_a, \lambda_b; t_0, x_0) \end{array} \tag{5}$$

is defined as a composite map $S = S_b \circ S_a$, to avoid discontinuity in the derivative of the solution at $t = t_0$, and $t = t_0 + \alpha\tau$, where S_a and S_b are given by the following submaps:

$$\begin{array}{rcl} S_a : R^n & \rightarrow & R^n \\ x_0 & \mapsto & x_1 = \varphi_a(t_0 + \alpha\tau, \lambda_0, \lambda_a; t_0, x_0), \end{array} \tag{6}$$

$$\begin{array}{rcl} S_b : R^n & \rightarrow & R^n \\ x_1 & \mapsto & x_2 = \varphi_b(t_0 + \tau, \lambda_0, \lambda_b; t_0 + \alpha\tau, x_1). \end{array} \tag{7}$$

In this way, the behavior of a periodic solution can be reduced to that of a fixed point on the Poincaré map.

The numerical determination of a bifurcation set is accomplished by using the method proposed by Kawakami [2] so that the accurate location of a fixed point and a bifurcation parameter value are calculated by solving the fixed point equation and the bifurcation condition simultaneously.

3. An Example of Bifurcation Analyses in Nonautonomous System Using BunKi

As an example of the bifurcation analyses in the nonautonomous system with square-wave pulse trains, we applied the method to a molecular circadian clock model with light-dark cycles [4] described as follows:

$$\begin{aligned} \frac{dM}{dt} &= v_s(t)\frac{K_I^n}{K_I^n + F_N^n} - v_m\frac{M}{K_M + M}, \\ \frac{dF_C}{dt} &= k_s M - v_d\frac{F_C}{K_d + F_C} - k_1 F_C + k_2 F_N, \\ \frac{dF_N}{dt} &= k_1 F_C - k_2 F_N, \end{aligned} \tag{8}$$

where M, F_C, and F_N are state variables, and $K_I, K_d, K_M, k_1, k_2, k_s, v_d$, and v_m are system parameters. Through this report, the values of system parameters in Eq. (8) are fixed as $n = 4, K_I = 1, K_d = 0.13, K_M = 0.5, k_1 = 0.5, k_2 = 0.6, k_s = 0.5, v_d = 1.4, v_m = 0.5$, and $\alpha = 0.5$. $v_s(t)$ is an external periodic input of square-wave pulse trains. In this model, quasi-periodic oscillations, chaos attractors, and many bifurcation phenomena can be observed. By using the software BunKi, we demonstrate an example of bifurcation analysis of periodic solution observed in the molecular circadian clock model with the light-dark cycles.

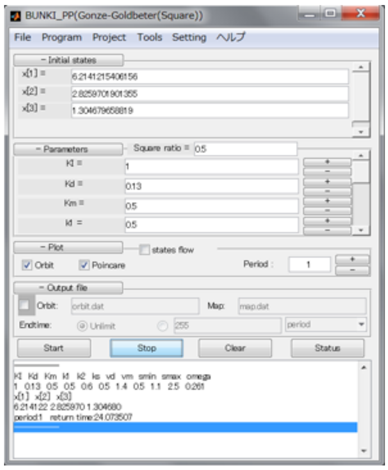


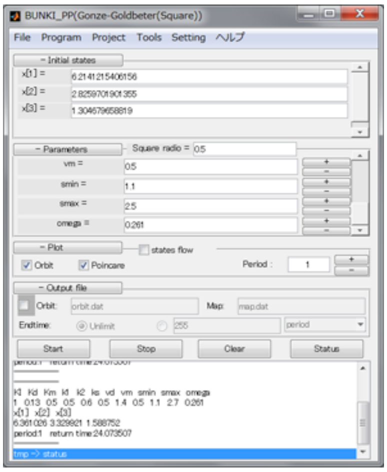
Figure 2: The system editor (SE).

3.1. Description of System Equations

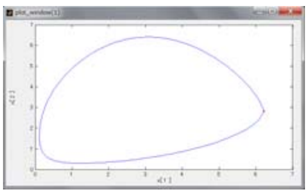
Figure 2 shows the application window of the system editor (SE). The system equations targeted for analysis is described through SE, and then several analysis tools is created. In SE, state variables of the non-autonomous system are described as $x[i]$ for $i = 1, 2, ...n$, where n is the dimension of the dynamical system. The system parameters are described by arbitrary names which is sandwiched by the symbols $ as shown in Fig. 2. If the system targeted for analysis is a non-autonomous system with square-wave pulse trains, select the button "Non-autonomous (square)" and describe $s(t)$ as the square-wave input into equations. When several analysis tools are created by SE, two parameters of s_{max} and s_{min} in $s(t)$ and an angular frequency of periodic force (ω) are automatically added as the system parameters. The ratio α is also added as a setting value with respect to the square-wave input.

3.2. Simulation and Search of the Fixed Point

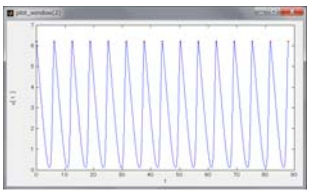
After several analysis tools is created by SE, we must conduct simulations to search the stable fixed point. Figure 3(a) shows the application window to conduct real-time simulations, called phase portrait (PP). The PP enables to draw the trajectory of the periodic solution on an arbitrary phase plane, the coordinate of the fixed point and time course of the periodic solution on a real-time basis. Initial values and the values of system parameters can set with the application window as shown in Fig.3(a). An example of the trajectory of the periodic solution observed in Eq. (8) and the time course are shown in Fig. 3(b). The value of system parameters can be also changed, and the coordinate value when the area on the plot window is clicked is used as a new initial value on a real-time basis.



(a)



(b)



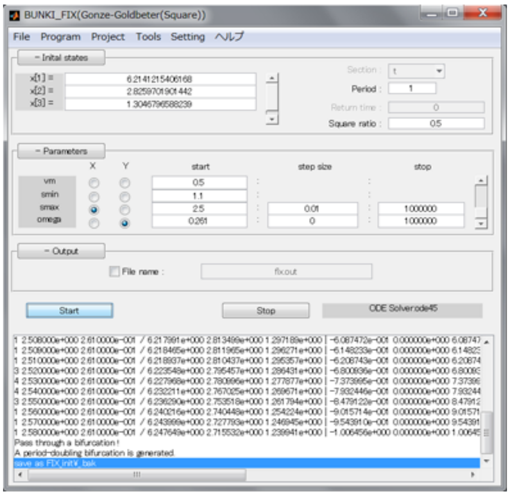
(c)

Figure 3: The real-time simulator, phase portrait (PP). (a) control window, (b) plot window of the trajectory of the periodic solution, and (c) time-course window.

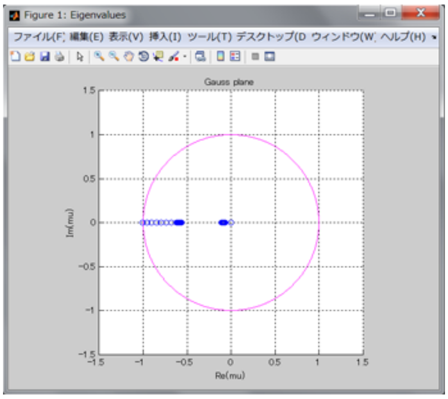
3.3. Detection of the Occurrence of the Bifurcation

Next, we evaluate the stability of the fixed point, and detect the parameter value when bifurcation occurs. Figure 4(a) shows the application window in a tool to detect the occurrence of bifurcations, called fixed point (FIX). To evaluate the stability of a fixed point and to detect the occurrence of the bifurcation, several information such as initial states of the fixed point, the values of system parameters, the period of periodic solution, etc., are required. Information related to the periodic solution can be taken by executing the export function of PP.

By using FIX, the eigenvalues of solution in a linearized system around the fixed point are calculated. Then the stability of the fixed point can be evaluated. The analysis tool, FIX, enables to know the stability of a fixed point by drawing the eigenvalue distribution on the plot window of eigenvalues as shown in Fig. 4(b). The eigenvalue distribution changes as the value of system parameter changes. Bifurcation occurs when one of the eigenvalues passes through the unit circle on the plot window in Fig. 4(b) by changing the value of system parameters. Figure 4(b) shows the example of an occurrence of the bifurcation that the eigenvalue passes through -1. Thus we can detect the occurrence of a period-doubling bifurcation of the fixed point.



(a)

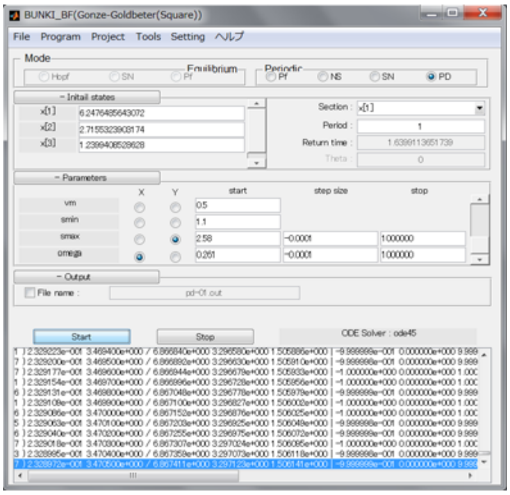


(b)

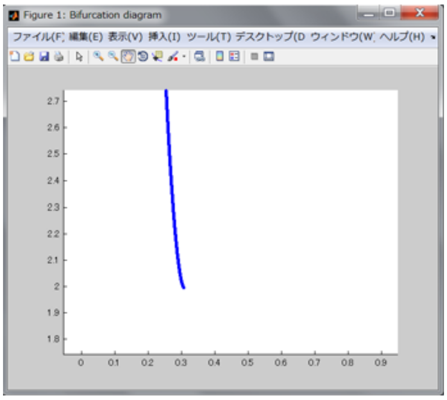
Figure 4: The calculating and detecting tool, fixed point (FIX). (a) control window, and (b) plot window to view the eigenvalue distribution.

3.4. Calculation of Bifurcation Curves

After a bifurcation point is detected by FIX, we conduct a calculation for the bifurcation sets on an arbitrary two parameter plane. Figure 5(a) shows the application window in a tool to calculate the bifurcation sets, called bifurcation (BF). Several information related to an initial point when the bifurcation occurs obtained by using FIX are imported to BF. Based on the initial point of the occurrence of the bifurcation, BF calculates next bifurcation point and traces continuously the bifurcation set on the arbitrary two parameter plane as shown in Fig. 5(b). The calculated result of the bifurcation set is drawn in the plot window on a real-time basis. At the same time, all eigenvalues are calculated to detect the occurrence of other bifurcations.



(a)

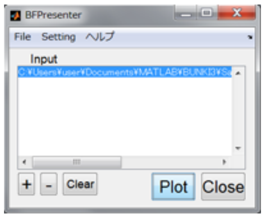


(b)

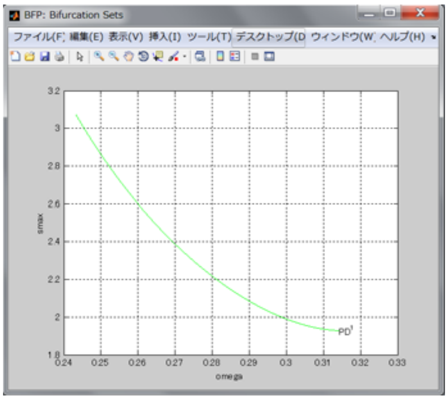
Figure 5: The tracking tool of bifurcation set, bifurcation (BF). (a) control window, and (b) the plot widow to view the bifurcation set on a real-time basis.

3.5. Bifurcation Diagram

When the bifurcation set is obtained, we need to check the bifurcation set on a two-parameter bifurcation diagram. Figure 6(a) shows the application window in a tool to draw a two-parameter bifurcation diagram, called BFPresenter. BFPresenter provide an interface to draw the two-parameter bifurcation diagram from output files created by BF. The bifurcation diagram that is made from BFPresenter is shown in Fig. 6(b).



(a)



(b)

Figure 6: The drawing tool of a two-parameter bifurcation diagram, called BFPresenter. (a) control window, and (b) the bifurcation diagram.

4. Conclusion

We have added a novel function that enabled the software to treat bifurcations of periodic solutions observed in a non-autonomous system with the square-wave pulse trains. Furthermore, we demonstrated an example of bifurcation analysis of periodic oscillation observed in a circadian clock model with light-dark cycles. As a future work, we will try to apply the new function to a variety of the non-autonomous system with square-wave pulses such as nervous systems, electric circuits, and social systems.

References

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