Convergence of the KP Solution of a Flanged Rectangular Waveguide

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1. Introduction

The radiation from an open-ended rectangular waveguide with a conducting flange is a classical problem and has been extensively studied by a variety of methods [1]–[5]. Since this structure appears in many EM problems such as the diffraction by single and coupled apertures in a thick screen and the radiation from slot antennas, an accurate solution of a flanged waveguide is important. To obtain the accurate result, the cross-polarized component and higher-order modes on the waveguide aperture must be considered in the formulation, and many authors have considered them. The inclusion of the edge property in the field is also effective in obtaining a highly accurate and fast convergent solution, but the solution derived by including the proper edge condition is rather sparse [3, 6]. Mongiardo and Rozzi [3] used the singular integral equation approach to derive the solution. Serizawa and Hongo [6] obtained the exact solution that satisfies the proper edge condition by applying the method of the Kobayashi potential (KP) [7].

In this paper, we numerically study the convergence of the KP solution of a flanged rectangular waveguide. The expressions of the physical quantity include unknown coefficients and they are determined by solving matrix equations whose elements consist of double infinite integrals and double infinite series that include four Bessel functions. These integral and series are calculated with the desired accuracy by applying the asymptotic approximation of the Bessel function. To verify the effect of inclusion of the edge property, the reflection coefficient and mode amplitudes in the waveguide are computed for various edge parameters and we show differences of convergence among the results.

2. Exact Solution of a Flanged Waveguide

Consider the radiation from a $2a \times 2b$ size rectangular waveguide terminated by an infinite conducting flange as described in Fig. 1. We assume that the half-space (region I) and the inside of the waveguide (region II) are filled with isotropic and homogeneous lossless mediums with parameters (ϵ_1, μ_1) and (ϵ_2, μ_2), respectively. The problem is to determine the field $\mathbf{E}^d$ radiated from the waveguide into region I and to evaluate the reflected wave $\mathbf{E}^r$ in region II, when the waveguide is excited by TE- and/or TM-modes ($\mathbf{E}^i$ means the incident wave). In this analysis, the harmonic time dependence $\exp(j\omega t)$ is assumed. Since the formulation that takes into account the proper edge condition is given in [6], we only show the results.

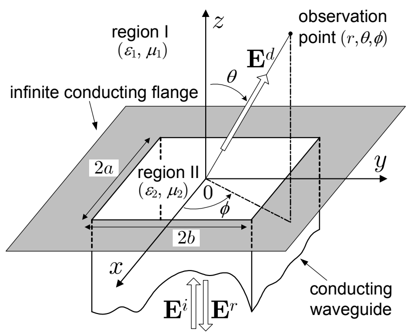

Figure 1: Radiation of an Electromagnetic Wave from a Flanged Rectangular Waveguide.

2.1 Fields in the Waveguide and Half-Space

The fields in the waveguide are represented by a linear combination of the TE- and TM-modal functions and axial components of vector potentials that satisfy the boundary conditions are given by

$$\text{TE mode:} \quad \begin{pmatrix} F_z^i \\ F_z^r \end{pmatrix} = a\epsilon_2 \sum_{\substack{m=0 \\ (m,n)\neq(0,0)}}^{\infty} \sum_{n=0}^{\infty} \begin{pmatrix} A_{mn}^{(E)} \exp(-jh_{mn}z_a) \\ B_{mn}^{(E)} \exp(jh_{mn}z_a) \end{pmatrix} \cos\frac{m\pi}{2}(\xi+1)\cos\frac{n\pi}{2}(\eta+1), \quad (1a)$$

$$\text{TM mode:} \quad \begin{pmatrix} A_z^i \\ A_z^r \end{pmatrix} = \frac{\kappa_2^2}{\omega} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{pmatrix} A_{mn}^{(M)} \exp(-jh_{mn}z_a) \\ B_{mn}^{(M)} \exp(jh_{mn}z_a) \end{pmatrix} \sin\frac{m\pi}{2}(\xi+1)\sin\frac{n\pi}{2}(\eta+1), \tag{1b}$$

$$h_{mn} = \sqrt{\kappa_2^2 - (m\pi/2)^2 - p^2(n\pi/2)^2}, \quad p = a/b\,(=1/q)\,, \quad \kappa_2 = k_2 a, \quad k_2 = \omega\sqrt{\epsilon_2\mu_2}, \tag{1c}$$

where $\xi = x/a$, $\eta = y/b$, $z_a = z/a$ are the normalized variables.

For the radiated waves in the half-space, we use the x and y components of the electric vector potential $\mathbf{F}$ and they are given by the Kobayashi potential for the present problem.

$$F_x^d = a\epsilon_1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{1}{\zeta(\alpha,\beta)}\Big\{\Lambda_{2m}^{\sigma}(\alpha)\cos\alpha\xi\Big[A_{mn}^{(x)}\Lambda_{2n}^{\tau}(\beta)\cos\beta\eta + B_{mn}^{(x)}\Lambda_{2n+1}^{\tau}(\beta)\sin\beta\eta\Big]$$
$$+\Lambda_{2m+1}^{\sigma}(\alpha)\sin\alpha\xi\Big[C_{mn}^{(x)}\Lambda_{2n}^{\tau}(\beta)\cos\beta\eta + D_{mn}^{(x)}\Lambda_{2n+1}^{\tau}(\beta)\sin\beta\eta\Big]\Big\}\exp\left[-\zeta(\alpha,\beta)\ z_a\ \right]d\alpha d\beta, \tag{2a}$$

$$F_y^d = a\epsilon_1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{1}{\zeta(\alpha,\beta)}\Big\{\Lambda_{2m}^{\tau}(\alpha)\cos\alpha\xi\Big[A_{mn}^{(y)}\Lambda_{2n}^{\sigma}(\beta)\cos\beta\eta + B_{mn}^{(y)}\Lambda_{2n+1}^{\sigma}(\beta)\sin\beta\eta\Big]$$
$$+\Lambda_{2m+1}^{\tau}(\alpha)\sin\alpha\xi\Big[C_{mn}^{(y)}\Lambda_{2n}^{\sigma}(\beta)\cos\beta\eta + D_{mn}^{(y)}\Lambda_{2n+1}^{\sigma}(\beta)\sin\beta\eta\Big]\Big\}\exp\left[-\zeta(\alpha,\beta)\ z_a\ \right]d\alpha d\beta, \tag{2b}$$

$$\Lambda_\ell^\nu(x) = J_{\ell+\nu}(x)/x^\nu, \quad \zeta(\alpha,\beta) = \sqrt{\alpha^2 + p^2\beta^2 - \kappa_1^2}, \quad \kappa_1 = k_1 a, \quad k_1 = \omega\sqrt{\epsilon_1\mu_1}, \tag{2c}$$

where σ and τ are selected so as to incorporate the edge property in the electric field [8]. Since the electric field behaves like $E_x^d \propto (1-\xi^2)^{\tau-\frac{1}{2}}(1-\eta^2)^{\sigma-\frac{1}{2}}$, $E_y^d \propto (1-\xi^2)^{\sigma-\frac{1}{2}}(1-\eta^2)^{\tau-\frac{1}{2}}$ near edges, we select $\sigma = 7/6$ and $\tau = 1/6 (= \sigma - 1)$ for the same medium parameters ($\epsilon_1 = \epsilon_2$, $\mu_1 = \mu_2$).

2.2 Matrix Equation

Matrix equations for the unknown expansion coefficients $A_{mn}^{(x)} \sim D_{mn}^{(x)}$ and $A_{mn}^{(y)} \sim D_{mn}^{(y)}$ are given by

$$\begin{bmatrix} K_{A(m,n,s,t)}^{(u,v)} + R_\mu S_{A(m,n,s,t)}^{(u,v)} & (-1)^{u+v}p\{G_{A(m,n,s,t)}^{(u,v)} + R_\mu T_{A(m,n,s,t)}^{(u,v)}\} \\ (-1)^{u+v}q\{G_{B(m,n,s,t)}^{(\bar u,\bar v)} + R_\mu T_{B(m,n,s,t)}^{(\bar u,\bar v)}\} & K_{B(m,n,s,t)}^{(\bar u,\bar v)} + R_\mu S_{B(m,n,s,t)}^{(\bar u,\bar v)} \end{bmatrix} \begin{bmatrix} X_{mn}^{(u,v)} \\ Y_{mn}^{(\bar u,\bar v)} \end{bmatrix}$$
$$= 2jR_\mu(-1)^{u+v}\begin{bmatrix} P_{st}^{(u,v)} \\ Q_{st}^{(\bar u,\bar v)} \end{bmatrix}, \qquad \begin{cases} (u,v) = (0,0),\ (0,1),\ (1,0),\ (1,1) \\ s = 0,1,2,\cdots, \quad t = 0,1,2,\cdots \end{cases}, \tag{3a}$$

$$X_{mn}^{(0,0)} = A_{mn}^{(x)}, \quad X_{mn}^{(0,1)} = B_{mn}^{(x)}, \quad X_{mn}^{(1,0)} = C_{mn}^{(x)}, \quad X_{mn}^{(1,1)} = D_{mn}^{(x)}, \tag{3b}$$
$$Y_{mn}^{(0,0)} = A_{mn}^{(y)}, \quad Y_{mn}^{(0,1)} = B_{mn}^{(y)}, \quad Y_{mn}^{(1,0)} = C_{mn}^{(y)}, \quad Y_{mn}^{(1,1)} = D_{mn}^{(y)}, \tag{3c}$$

where $\bar u = 1-u$, $\bar v = 1-v$, and $R_\mu = \mu_1/\mu_2$. The other symbols are defined by

$$P_{st}^{(u,v)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\left(\tfrac{2m+1+u}{2}\pi\right)h_{2m+1+u,2n+v}A_{2m+1+u,2n+v}^{(E)}\Lambda_{2s+u}^{\tau'}\left(\tfrac{2m+1+u}{2}\pi\right)\Lambda_{2t+v}^{\sigma'}\left(\tfrac{2n+v}{2}\pi\right)$$
$$+ p\kappa_2^2\sum_{m=0}^{\infty}\sum_{n=1-v}^{\infty}(-1)^{m+n}\left(\tfrac{2n+v}{2}\pi\right)A_{2m+1+u,2n+v}^{(M)}\Lambda_{2s+u}^{\tau'}\left(\tfrac{2m+1+u}{2}\pi\right)\Lambda_{2t+v}^{\sigma'}\left(\tfrac{2n+v}{2}\pi\right), \tag{4a}$$

$$Q_{st}^{(u,v)} = q\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\left(\tfrac{2n+1+v}{2}\pi\right)h_{2m+u,2n+1+v}A_{2m+u,2n+1+v}^{(E)}\Lambda_{2s+u}^{\sigma'}\left(\tfrac{2m+u}{2}\pi\right)\Lambda_{2t+v}^{\tau'}\left(\tfrac{2n+1+v}{2}\pi\right)$$
$$- q^2\kappa_2^2\sum_{m=1-u}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\left(\tfrac{2m+u}{2}\pi\right)A_{2m+u,2n+1+v}^{(M)}\Lambda_{2s+u}^{\sigma'}\left(\tfrac{2m+u}{2}\pi\right)\Lambda_{2t+v}^{\tau'}\left(\tfrac{2n+1+v}{2}\pi\right). \tag{4b}$$

The matrix elements consist of double infinite integrals and double infinite series.

$$K_{A(m,n,s,t)}^{(u,v)} = \int_0^{\infty}\int_0^{\infty}\frac{\kappa_1^2-\alpha^2}{\zeta(\alpha,\beta)}\Lambda_{2m+u}^{\sigma}(\alpha)\Lambda_{2s+u}^{\tau'}(\alpha)\Lambda_{2n+v}^{\tau}(\beta)\Lambda_{2t+v}^{\sigma'}(\beta)d\alpha d\beta, \tag{5a}$$

$$G_{A(m,n,s,t)}^{(u,v)} = \int_0^{\infty}\int_0^{\infty}\frac{\alpha\beta}{\zeta(\alpha,\beta)}\Lambda_{2m+1-u}^{\tau}(\alpha)\Lambda_{2s+u}^{\tau'}(\alpha)\Lambda_{2n+1-v}^{\sigma}(\beta)\Lambda_{2t+v}^{\sigma'}(\beta)d\alpha d\beta, \tag{5b}$$

$$S_{A(m,n,s,t)}^{(u,v)} = \pi^2\sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty}\frac{\kappa_2^2 - \left(\frac{2m'+1+u}{2}\pi\right)^2}{(1+\delta_{0,2n'+v})\gamma_{2m'+1+u,2n'+v}}$$

$$\times \Lambda^{\sigma}_{2m+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\tau'}_{2s+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\tau}_{2n+v}\left(\tfrac{2n'+v}{2}\pi\right)\Lambda^{\sigma'}_{2t+v}\left(\tfrac{2n'+v}{2}\pi\right), \tag{6a}$$

$$T^{(u,v)}_{A(m,n,s,t)} = \pi^2 \sum_{m'=0}^{\infty}\sum_{n'=0}^{\infty} \frac{1}{\gamma_{2m'+1+u,2n'+v}} \times \Lambda^{\tau}_{2m+1-u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\tau'}_{2s+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\sigma}_{2n+1-v}\left(\tfrac{2n'+v}{2}\pi\right)\Lambda^{\sigma'}_{2t+v}\left(\tfrac{2n'+v}{2}\pi\right), \tag{6b}$$

where $\delta_{\ell\ell'}$ is the Kronecker delta and $\gamma_{mn} = \sqrt{(m\pi/2)^2 + p^2(n\pi/2)^2 - \kappa_2^2} = jh_{mn}$. Parameters σ' and τ' are determined by considering the edge property of the magnetic field [8]. The magnetic field behaves like $H_x^d \propto (1-\xi^2)^{\frac{1}{2}-\tau'}(1-\eta^2)^{\frac{1}{2}-\sigma'}$, $H_y^d \propto (1-\xi^2)^{\frac{1}{2}-\sigma'}(1-\eta^2)^{\frac{1}{2}-\tau'}$ near edges. For the same medium parameters, we select $\sigma' = -1/6 (= 1-\sigma)$ and $\tau' = 5/6 (= 1-\tau)$. However, to verify the effect of the edge property of magnetic field, we add an integral parameter $\nu (= 0, 1, 2, \ldots)$ to σ' and τ' ($\nu = 0$ is the exact case).

2.3 Reflection Coefficient and Mode Amplitudes

The reflection coefficients of the incident wave and the amplitudes of the higher-order modes can be obtained from $B^{(E)}_{mn}$ and $B^{(M)}_{mn}$. The expressions of $B^{(E)}_{mn}$ and $B^{(M)}_{mn}$ are given by

$$B^{(E)}_{mn} = \frac{1}{\lambda^2_{mn}}\left[\left(\frac{m\pi}{2}\right)F_{mn} + p\left(\frac{n\pi}{2}\right)E_{mn}\right] - A^{(E)}_{mn}, \tag{7a}$$

$$B^{(M)}_{mn} = \frac{1}{\lambda^2_{mn}h_{mn}}\left[\left(\frac{m\pi}{2}\right)E_{mn} - p\left(\frac{n\pi}{2}\right)F_{mn}\right] + A^{(M)}_{mn}, \tag{7b}$$

where $\lambda^2_{mn} = (m\pi/2)^2 + p^2(n\pi/2)^2$. E_{mn} and F_{mn} are defined by

$$E_{2m'+u,2n'+1+v} = -\frac{(-1)^{m'+n'+u+v}\pi^2}{1+\delta_{0,2m'+u}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Y^{(u,v)}_{mn}\Lambda^{\tau}_{2m+u}\left(\tfrac{2m'+u}{2}\pi\right)\Lambda^{\sigma}_{2n+v}\left(\tfrac{2n'+1+v}{2}\pi\right), \tag{8a}$$

$$F_{2m'+1+u,2n'+v} = -\frac{(-1)^{m'+n'+u+v}\pi^2}{1+\delta_{0,2n'+v}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X^{(u,v)}_{mn}\Lambda^{\sigma}_{2m+u}\left(\tfrac{2m'+1+u}{2}\pi\right)\Lambda^{\tau}_{2n+v}\left(\tfrac{2n'+v}{2}\pi\right). \tag{8b}$$

$E_{2m'+1+u,2n'+v}$ and $F_{2m'+u,2n'+1+v}$ are derived from (8a) and (8b).

3. Numerical Results and Discussion

To obtain numerical results for the physical quantities of interest, the matrix elements must be calculated. Since the elements consist of double infinite integrals and double infinite series, we apply an effective method of computing them [5], [6] and the integrals and series are split into six and five parts, respectively. Part of the integrals and series that include infinite regions are transformed into more simple forms by using the asymptotic approximations of their integrands and summands. Thus the integrals and series are computed with the desired accuracy. In practical computation, the matrix equation is truncated to a finite size. When the maximum value of m, n, s, and t in (3) is $n_k - 1$, the matrix size is $2n_k^2 \times 2n_k^2$. By changing the value of n_k, we can numerically verify the convergence of the solution. To simplify the computation, we consider the case that the mediums I and II have the same parameters, that is, $\epsilon_1 = \epsilon_2 = \epsilon$, $\mu_1 = \mu_2 = \mu$, and $k_1 = k_2 = k$ ($\kappa_1 = \kappa_2 = \kappa$). We compute the reflection coefficient and amplitudes of higher order modes for $ka = 2.235$ and $a/b = 2.25$ (WR90 at a frequency of 9.33GHz) when the waveguide is excited by TE_{10} mode ($A^{(E)}_{10} = 1$). Figure 2 shows the results of the modulus and phase of the reflection coefficient $B^{(E)}_{10}$ and the plots are obtained for four kinds of edge properties. The property of $\sigma = 1$ and $\tau = 0$ is the same as that of the thin plate ($\sigma' = 1$ and $\tau' = 1$ are selected in [5]). It is found from the figures that the rate of convergence for the correct edge property ($\nu = 0$) is much faster than others, but all the results converge to the same value (the convergence rate depends on the edge property incorporated into the solution). Figure 3 shows the results of higher order modes $B^{(E)}_{30}, B^{(E)}_{12}$. We also find that the rate of convergence of the exact solution is much faster than others.

4. Conclusion

By usign the exact solution of a flanged rectangular waveguide based on the Kobayashi potential, the numerical result of the reflection coefficient and the amplitudes of higher order modes was obtained

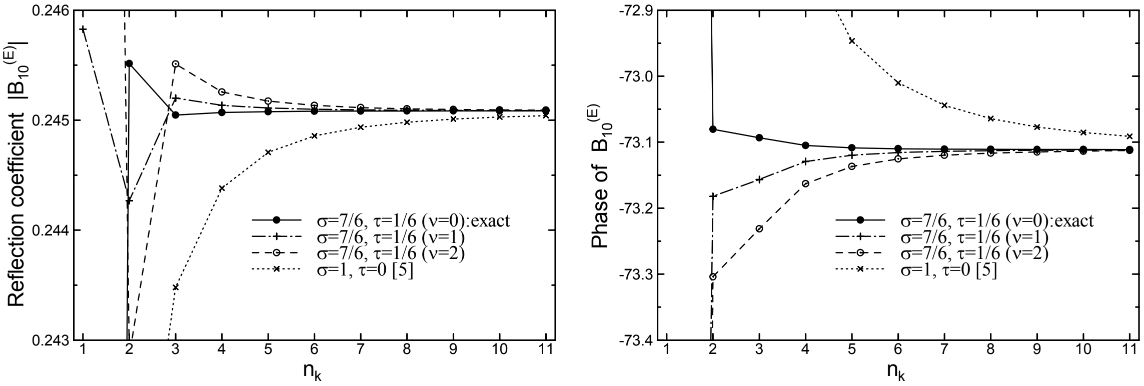

Figure 2: Modulus and Phase of Reflection Coefficient for $ka = 2.235$ and $a/b = 2.25$.

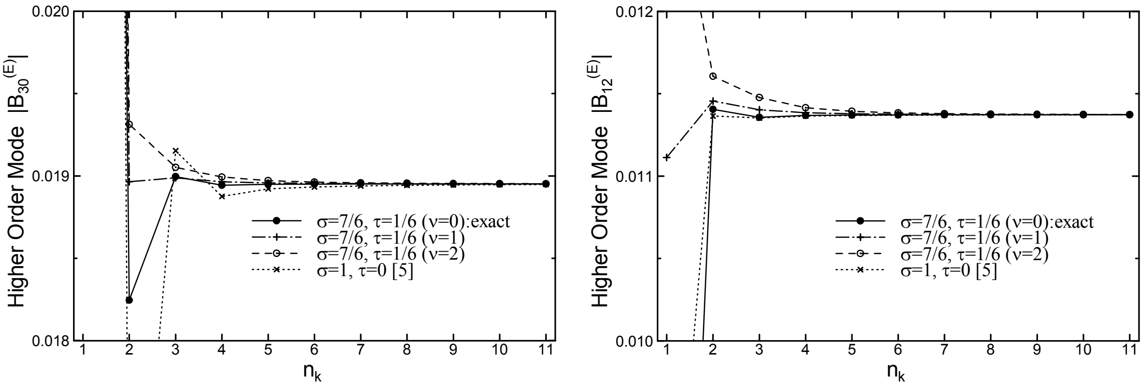

Figure 3: Amplitudes of Higher Order Modes $|B_{30}^{(E)}|$ and $|B_{12}^{(E)}|$ Corresponding to Fig. 2.

and we numerically verified the effect of inclusion of the correct edge property on the convergence of the solution. As a result, it was shown that the edge property about magnetic field is also indispensable for obtaining a highly accurate and fast convergent solution as well as that about electric field.

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