Rediscovering Stability Criterion of Yee's FDTD Scheme for One-Dimensional Infinite Homogeneous Conductive Space Using a Multidimensional Signal Processing Theorem

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Abstract

The Yee's FDTD method for homogeneous conductive space is treated as a nonsymmetric half-plane (NSHP) support filter. Its stability criterion is established from a modified multidimensional signal processing theorem. The result agrees with that obtained by the von Neumann's method. This new approach may be extended for other FDTD-like methods.

Keywords : Stability, FDTD methods, Multidimensional signal processing, NSHP filters

1. Introduction

The FDTD method [1][2] and its variations have been widely used. The stability of these methods are often analyzed using von Neumann's method [3][4]. On the other hand, these numerical schemes can be also viewed from a new aspect as space-time digital signal processing. Their stability should be consistent with the stability criteria used in multidimensional signal processing. This paper is to show that the stability criteria for the 1-D Yee's FDTD methods for homogeneous conductive space [5] can be rediscovered from the stability criteria for nonsymmetric half-plane (NSHP) support filters developed by Ekstrom and Woods [6][7] in multidimensional signal processing.

Many stability criteria for multidimensional signal processing have been proposed [7]; however, most of them deal with filters which are causal in all dimensions, such as the criteria based or generalized on theorems proposed by Shanks et al. [8] or Huang [9], and the criteria for N-dimensional filters by Justice and Shanks [10]. It will be shown in this paper that the space-time filter corresponding to Yee's FDTD method for 1-D conductive space cannot be put in standard rational forms, which are necessary to apply the stability criterion for filters causal in all dimensions.

In addition, all the stability criteria in digital signal processing treat the condition that the poles of the multidimensional z-transform of the system transfer functions are on the unit circle as a cause of instability. However, the von Neumann's method regards its equivalent condition as stable in numerical analysis, since the field will not grow in this case. We will introduce the concept of exponential stability such that results obtained via multidimensional signal processing theorems can be consistent with the von Neumann's method.

This paper starts with the introduction of the stability criteria for NSHP support filters in Section 2. Then we introduce the concept of exponential stability and modify the stability criteria for NSHP support filters in Section 3. The modified criteria are then applied to rediscover the stability criteria of Yee's FDTD for 1-D homogeneous conductive space, in Sections 4, respectively. Finally, the conclusion will be given in Section 5.

2. Stability Criteria for NSHP Support Filters

An NSHP region is defined as [7]

$$\text{NSHP Region} \equiv \{n_1 \ge 0, n_2 \ge 0\} \cup \{n_1 < 0, n_2 > 0\}\,. \tag{1}$$

A filter defined on the NSHP support has been shown [6] that the filter is stable in the bounded-input bounded-output (BIBO) sense if the system function $\mathrm{H}(\mathrm{z}_1, \mathrm{z}_2)$ satisfies the following conditions:

(a) $\mathrm{H}(\mathrm{z}_1, \infty)$ is analytic, i.e., free of singularities, on $\{|\mathrm{z}_1| \ge 1\}$.

(b) $\mathrm{H}(e^{j\Omega_1}, \mathrm{z}_2)$ is analytic on $\{|\mathrm{z}_2| \ge 1\}$, for all $\Omega_1 \in [-\pi, +\pi]$.

3. Exponential Stability

By the criteria (a) and (b) introduced in Section II, the system is unstable if there exists poles on the unit circle $|\mathrm{z}_1| = 1$ or $|\mathrm{z}_2| = 1$. However, under condition (b) it can be shown that the poles of the system function for the Yee's FDTD scheme in free space are exactly on the unit circle when it is regarded as stable by FDTD researchers. Thus, we have to apply the concept of exponential stability.

By exponential stability, we assume that the system function is written as $\mathrm{H}(\mathrm{z}_1, \mathrm{z}_2) = \lim_{a \to 0^+} \lim_{b \to 0^+} \mathrm{H}(e^{-a}\mathrm{z}_1, e^{-b}\mathrm{z}_2)$, and then the stable conditions become

(a') $\mathrm{H}(\mathrm{z}_1, \infty)$ is analytic, i.e., free of singularities, on $\{\lim_{a \to 0^+} |e^{-a}\mathrm{z}_1| \ge 1\}$, or $\{|\mathrm{z}_1| > 1\}$.

(b') $\mathrm{H}(e^{j\Omega_1}, \mathrm{z}_2)$ is analytic on $\{\lim_{b \to 0^+} |e^{-b}\mathrm{z}_2| \ge 1\}$, or $\{|\mathrm{z}_2| > 1\}$ for all $\Omega_1 \in [-\pi, +\pi]$.

4. Yee's FDTD Algorithm for Conductive Space

Yee's partial difference equations for 1-D homogeneous conductive space are

$$\eta_0 H_y^{n+1/2}[k + \frac{1}{2}] = \eta_0 H_y^{n-1/2}[k + \frac{1}{2}] - s\{E_x^n[k+1] - E_x^n[k]\}\,, \tag{2}$$

$$\begin{aligned}\left(1 + \frac{\Delta t}{2\tau}\right) E_x^{n+1}[k] &= \left(1 - \frac{\Delta t}{2\tau}\right) E_x^n[k] - \\ &\quad \frac{s}{\varepsilon_\infty}\left\{\eta_0 H_y^{n+1/2}[k + \frac{1}{2}] - \eta_0 H_y^{n+1/2}[k - \frac{1}{2}]\right\}\end{aligned}\,. \tag{3}$$

Here η_0 is the intrinsic impedance of the free space, s is the Courant number, $\tau = \dfrac{\varepsilon_0 \varepsilon_\infty}{\sigma}$, ε_0 is the permittivity of the free space, ε_∞ and σ are the relative permittivity and the conductivity of the space, respectively, and Δt is the sampling time interval. Since the stability in numerical analysis is defined as that the initial conditions will not grow without bound in later iterations, we need not have excitation sources in (2)(3).

Define the two-dimensional z-transforms

$$E_x(\zeta, \mathrm{z}) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} E_x^n[k] \mathrm{z}^{-n} \zeta^{-k}\,, \tag{4}$$

$$\eta_0 H_y(\zeta, \mathrm{z}) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \eta_0 H_y^{n+1/2}[k + \frac{1}{2}] \mathrm{z}^{-(n+1/2)} \zeta^{-(k+1/2)}\,. \tag{5}$$

If we set initial conditions $E_x^0[k] = 0$ for $k < 0$, and $\eta_0 H_y^{-1/2}[k] = 0$ for all k, equations (2)(3) are obviously defined on an NSHP region (1), with n_1 and n_2 corresponding to the spatial index k and the time index n, respectively. Accordingly, ζ and z are the z_1 and z_2, respectively, in Sections 2 and 3.

Take the two-dimensional z-transform of (2) and (3), we can solve

$$\mathrm{H}(\zeta,\mathrm{z}) = \frac{E_x(\zeta,\mathrm{z})}{E_x^0(\zeta)}$$

$$= \frac{\zeta(\mathrm{z}-1)\mathrm{z}\left(1+\frac{\Delta t}{2\tau}\right)}{\zeta(1+\frac{\Delta t}{2\tau})\mathrm{z}^2 - 2\zeta z + \frac{s^2}{\varepsilon_\infty}(\zeta-1)^2\mathrm{z} + \zeta(1-\frac{\Delta t}{2\tau})} . \qquad (6)$$

where, $E_x^0(\zeta) = \sum_{k=0}^{\infty} E_x^0[k]\zeta^{-k}$. Since $\mathrm{H}(\zeta,\infty) \to 1$, stability criterion (a') is satisfied.

Next, we check criterion (b') by considering $\mathrm{H}(e^{jK},\mathrm{z})$ and find that the poles satisfy

$$\left(1+\frac{\Delta t}{2\tau}\right)\mathrm{z}^2 - 2(1-2\nu^2)\mathrm{z} + \left(1-\frac{\Delta t}{2\tau}\right) = 0, \qquad (7)$$

where $\nu^2 = \frac{s^2}{\varepsilon_\infty}\sin^2\frac{K}{2}$, and (7) is identical with the stability polynomial equation obtained via the von Neumann's method in [5].

The properties of the roots of (7) depend on the discriminator $\Delta = \left(1-2\nu^2\right)^2 + \left(\frac{\Delta t}{2\tau}\right)^2 - 1$. When

(i) $\frac{\Delta t}{2\tau} > 1$ or

(ii) $\frac{\Delta t}{2\tau} < 1$ with $\nu^2 \le \frac{1-\sqrt{1-\left(\frac{\Delta t}{2\tau}\right)^2}}{2}$ or $\nu^2 \ge \frac{1+\sqrt{1-\left(\frac{\Delta t}{2\tau}\right)^2}}{2}$,

the discriminator is larger than or equal to zero, and (7) will have two real roots α,β. To satisfy criterion (b'), we need $-1 \le \alpha,\beta \le 1$, which leads to $-2 \le \alpha+\beta \le 2$, $(\alpha+1)(\beta+1) \ge 0$, $(\alpha-1)(\beta-1) \ge 0$, and results in $\nu^2 \le 1+\frac{\Delta t}{4\tau}$, $\nu^2 \ge 0$, $\nu^2 \le 1$, respectively. On the other hand, when $\frac{\Delta t}{2\tau} < 1$ with $\frac{1-\sqrt{1-\left(\frac{\Delta t}{2\tau}\right)^2}}{2} < \nu^2 < \frac{1+\sqrt{1-\left(\frac{\Delta t}{2\tau}\right)^2}}{2}$, the discriminator is negative. The roots will be a complex conjugate pair, such as $\alpha \pm j\beta$, with α,β real. To satisfy criterion (b'), we require $\alpha^2+\beta^2 = (\alpha+j\beta)(\alpha-j\beta) < 1$, i.e., $0 < \frac{1-\frac{\Delta t}{2\tau}}{1+\frac{\Delta t}{2\tau}} < 1$, which is always true under the assumption of $0 < \frac{\Delta t}{2\tau} < 1$.

By combining all the above stability requirements, we achieve a single condition, $0 \le \nu^2 \le 1$. Since $\nu^2 = \frac{s^2}{\varepsilon_\infty}\sin^2\frac{K}{2}$, and $-\pi \le K \le \pi$, we have $0 < s \le \sqrt{\varepsilon_\infty}$, which is just the stability requirement for the Courant number in an infinite dielectric space without conductive loss. This result is also consistent with that given in [5].

5. Conclusions

We have introduced the stability criteria for NSHP supported two-dimensional filters, which was set up for multidimensional signal processing. Such criteria have been modified by exponential stability such that the system is regarded stable when poles locate on the unit circle. The modified criteria have been applied to rediscover the stability criteria for Yee's FDTD method for conductive space. The stability criterion (b') is found equivalent to the requirement derived from the von Neumann's method that the stability polynomial equation must be with roots within or on the unit circle. The stability criterion (a') was not found in literatures of numerical solution for partial differential equations before. However, they are related to just initial conditions, and are satisfied for the cases we deal with. One interesting question would be if these criteria can be generalized to 2-D or 3-D FDTD-family schemes. Since the von Neumann's method has been successively applied to those problems, it is very likely that the concept of stability criteria for NSHP supported filters can be generalized to *N*-dimensional. The problem is how to find a solid mathematical proof like those given in [6]. Another possible way is to reduce the three-dimensional z-transform solution to a two-dimensional one, and then apply the NSHP criteria given in this paper. These stability criteria in multidimensional signal processing, could be also used as an alternative derivation of stability criteria for other existent or new FDTD-like numerical schemes in homogeneous media, such as ADI in free space [11] and as a tool for validation of the common von-Neumann's method.

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