Coupled-Mode Formulation of Two-Parallel Photonic Crystal Waveguides

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Abstract

The coupled-mode formulation for two-dimensional coupled photonic crystal waveguides is discussed. Using a perturbation theory, the self-contained first-order coupled-mode equations are derived, which govern the evolution of the amplitudes of the eigenmodes of individual photonic crystal waveguides in isolation.

Keywords : Photonic crystal waveguides Coupled-mode equations Perturbation theory

1. Introduction

A photonic crystal waveguide (PCW) consists of a line defect introduced in a perfect photonic crystal. If two photonic crystal waveguides (PCWs) are placed in close proximity, a coupled PCW is formed and the optical power is efficiently transferred from one PCW to another. Recently, the coupled PCW has received much attention because of their promising applications to miniaturized photonic devices such as filters, switches, power dividers, and couplers. The optical propagation in the coupled PCWs has been extensively analyzed using the plane wave expansion method [1],[2], the finite difference time domain method [1],[3], the scattering matrix method combined with the lattice sums [4],[5], and the coupled-mode theory [1],[2],[3]. Among others, the coupled-mode theory is an analytical method based on a perturbation theory and gives an approximate solution, which enables ones to get a clear picture on the power transfer between two PCWs. However, the previous pertinent studies have mostly assumed a prescribed form of the coupled-mode equations and calculated the coupling coefficients using the propagation constants of *even* and *odd* modes obtained by other numerical methods.

In this paper, we present a self-contained formulation of the coupled-mode theory for coupled two-dimensional PCWs. Using a perturbation theory, the first-order coupled-mode equations are systematically derived, which govern the evolution of the modal amplitudes in individual PCWs. The mutual- and self-coupling coefficients are obtained in terms of the propagation constants and eigenmode solutions of the two PCWs in isolation.

2. Basic Equations

Let us consider the coupled two-dimensional PCWs consisting of a five-layered structure as shown in Fig. 1(a). The guiding layers "*a*" and "*b*" are separated by the barrier layer of photonic crystal. We do not specify here the particular configuration of three photonic crystal layers except that they have a common lattice constant h_z in the *z* direction. The upper photonic crystal, the lower photonic crystal, and the barrier are denoted by the subscripts "*U*", "*L*", and "*B*", respectively.

The guided mode in the coupled waveguides shown in Fig. 1(a) is expanded in terms of the Floquet modes as follows:

$$\psi(x,z) = \sum_{m=-\infty}^{\infty} (c_m^+ \mathrm{e}^{i\gamma_m x} + c_m^- \mathrm{e}^{-i\gamma_m x}) \mathrm{e}^{i\beta_m z} \tag{1}$$

where $\beta_m = \beta + 2m\pi / h_z$, $\gamma_m = \sqrt{k^2 - \beta_m^2}$, k is the wavenumber in the background medium, and β

is the mode propagation constant. We define by $\mathbf{c}^+$ and $\mathbf{c}^-$ the column vectors whose elements are the expansion coefficients c_m^+ and c_m^-. Note that $\mathbf{c}^+$ and $\mathbf{c}^-$ represent the amplitude vectors of the up-going and down-going Floquet modes. Let us denote by $\mathbf{a}^+$ and $\mathbf{a}^-$ the amplitude vectors of the Floquet modes defined at the upper and lower interfaces of the guiding layer "*a*" and by $\mathbf{b}^+$ and $\mathbf{b}^-$ those of the guiding layer "*b*". Applying a ray tracing to the Floquet modes, we have the following relations between four amplitude vectors:

$$\mathbf{a}^+ = \mathbf{\Lambda}_a(\beta)[\overline{\overline{\mathbf{R}}}_B(\beta)\cdot\mathbf{a}^- + \overline{\overline{\mathbf{T}}}_B(\beta)\cdot\mathbf{b}^+], \quad \mathbf{a}^- = \mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_U(\beta)\cdot\mathbf{a}^+ \tag{2}$$

$$\mathbf{b}^- = \mathbf{\Lambda}_b(\beta)[\overline{\overline{\mathbf{T}}}_B(\beta)\cdot\mathbf{a}^- + \overline{\overline{\mathbf{R}}}_B(\beta)\cdot\mathbf{b}^+], \quad \mathbf{b}^+ = \mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_L(\beta)\cdot\mathbf{b}^- \tag{3}$$

with

$$\mathbf{\Lambda}_a(\beta) = \left[\mathrm{e}^{i\gamma_m w_a}\,\delta_{mn}\right], \quad \mathbf{\Lambda}_b(\beta) = \left[\mathrm{e}^{i\gamma_m w_b}\,\delta_{mn}\right] \tag{4}$$

where $\overline{\overline{\mathbf{R}}}_U(\beta)$ is the generalized reflection matrix of the upper photonic crystal viewed from the guiding layer "*a*", $\overline{\overline{\mathbf{R}}}_L(\beta)$ is the generalized reflection matrix of the lower photonic crystal viewed from the guiding layer "*b*", and $\overline{\overline{\mathbf{R}}}_B(\beta)$ and $\overline{\overline{\mathbf{T}}}_B(\beta)$ are the generalized reflection and transmission matrices of the photonic crystal barrier between two guiding layers "*a*" and "*b*". If the lattice constants in the *z* and *x* directions, the radius and material constants of lattice elements, and the number of layers are specified for each of photonic crystals, the generalized reflection matrix $\overline{\overline{\mathbf{R}}}_p$ $(p = U, L, B)$ and transmission matrix $\overline{\overline{\mathbf{T}}}_p$ $(p = U, L, B)$ can be calculated [5] using the T-matrix of the isolated single circular rod and the lattice sums.

To keep the equations for the coupled waveguide system in symmetric form, we employ $\mathbf{a}^-$ and $\mathbf{b}^+$ as the leading amplitude vectors. Eliminating $\mathbf{a}^+$ from Eq.(2), we have

$$[\mathbf{I} - \mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_U(\beta)\mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_B(\beta)]\cdot\mathbf{a}^- = \mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_U(\beta)\mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{T}}}_B(\beta)\cdot\mathbf{b}^+ . \tag{5}$$

In the same way, from Eq.(3) we have

$$[\mathbf{I} - \mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_L(\beta)\mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_B(\beta)]\cdot\mathbf{b}^+ = \mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_L(\beta)\mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{T}}}_B(\beta)\cdot\mathbf{a}^- . \tag{6}$$

The mode propagation constant β and mode field distribution of the coupled waveguide system may be obtained [5] by directly solving the coupled linear equations (5) and (6). However, the transmission of the Floquet modes through the barrier with $\overline{\overline{\mathbf{T}}}_B(\beta)$ is typically small enough. In this case, we can assume a concept of weak coupling and approximate Eqs.(5) and (6) in the form of coupled-mode equations.

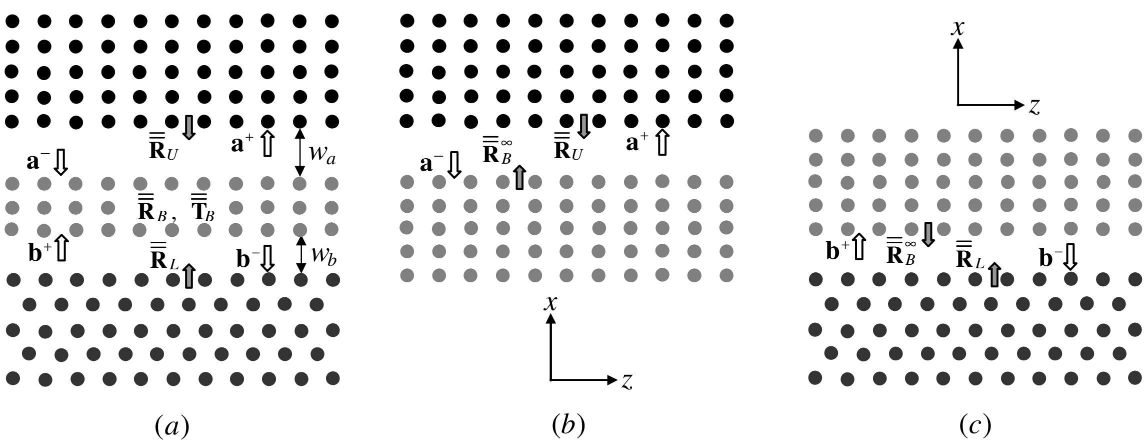

Fig. 1. Schematics of the coupled two-parallel photonic crystal waveguides: (*a*) coupled photonic crystal waveguide system, (*b*) isolated photonic crystal waveguide "*a*", and (*c*) isolated photonic crystal waveguide "*b*". The structures are two-dimensional.

3. Perturbation Theory

To perform a perturbation analysis, Eqs.(5) and (6) are rewritten in the following form:

$$\mathbf{D}_a(\beta)\cdot\mathbf{a}^- = \mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_U(\beta)\mathbf{\Lambda}_a(\beta)[\overline{\overline{\mathbf{R}}}_B(\beta)-\overline{\overline{\mathbf{R}}}_B^\infty(\beta)]\cdot\mathbf{a}^- + \mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_U(\beta)\mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{T}}}_B(\beta)\cdot\mathbf{b}^+ \quad (7)$$

$$\mathbf{D}_b(\beta)\cdot\mathbf{b}^+ = \mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_L(\beta)\mathbf{\Lambda}_b(\beta)[\overline{\overline{\mathbf{R}}}_B(\beta)-\overline{\overline{\mathbf{R}}}_B^\infty(\beta)]\cdot\mathbf{b}^+ + \mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_L(\beta)\mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{T}}}_B(\beta)\cdot\mathbf{a}^- \quad (8)$$

with

$$\mathbf{D}_a(\beta) = \mathbf{I} - \mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_U(\beta)\mathbf{\Lambda}_a(\beta)\overline{\overline{\mathbf{R}}}_B^\infty(\beta) \quad (9)$$

$$\mathbf{D}_b(\beta) = \mathbf{I} - \mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_L(\beta)\mathbf{\Lambda}_b(\beta)\overline{\overline{\mathbf{R}}}_B^\infty(\beta) \quad (10)$$

where $\overline{\overline{\mathbf{R}}}_B^\infty(\beta)$ is the generalized reflection matrix of the barrier with a thickness large enough so that two waveguides are isolated and $\overline{\overline{\mathbf{T}}}_B^\infty(\beta) \simeq \mathbf{0}$. As the barrier thickness increases, $\overline{\overline{\mathbf{R}}}_B(\beta)$ tends to $\overline{\overline{\mathbf{R}}}_B^\infty(\beta)$ while $\overline{\overline{\mathbf{T}}}_B(\beta)$ tends to zero. Under the weak coupling situation, we may regard the right hand sides of Eqs.(7) and (8) as a small perturbation. To find a solution through a perturbation analysis, the mode propagation constant and the amplitude vectors are expressed as follows:

$$\beta = \beta_0 + \delta\beta, \quad \mathbf{a}^- = \mathbf{a}_0 + \delta\mathbf{a}, \quad \mathbf{b}^+ = \mathbf{b}_0 + \delta\mathbf{b} \quad (11)$$

where $\delta\beta$, $\delta\mathbf{a}$, and $\delta\mathbf{b}$ denote the small perturbations with the order of $\overline{\overline{\mathbf{R}}}_B(\beta)-\overline{\overline{\mathbf{R}}}_B^\infty(\beta)$ and $\overline{\overline{\mathbf{T}}}_B(\beta)$ due to the finite thickness of the barrier. Equation (11) is substituted into Eqs.(7) and (8) to derive the leading order equations for the first-order analysis. For the zero-order analysis, we have the following equations:

$$\mathbf{D}_a(\beta_0)\cdot\mathbf{a}_0 = \mathbf{0}, \quad \mathbf{D}_b(\beta_0)\cdot\mathbf{b}_0 = \mathbf{0}. \quad (12)$$

Equation (12) yields the eigenmode solutions of each of waveguides "*a*" and "*b*" in isolation as shown in Figs 1(b) and (c). The mode propagation constants $\beta_0 = \beta_a$ for waveguide "*a*" and $\beta_0 = \beta_b$ for waveguide "*b*" are obtained [5] as the roots of $\det[\mathbf{D}_a(\beta_a)] = 0$ and $\det[\mathbf{D}_b(\beta_b)] = 0$, respectively. The associated solutions to $\mathbf{a}_0$ and $\mathbf{b}_0$ determine the amplitude coefficients of the Floquet modes. Using the zero-order solutions (12), the first-order perturbation equations are expressed in the following form:

$$\mathbf{D}_a(\beta_a)\cdot\delta\mathbf{a} = -\frac{\partial\mathbf{D}_a(\beta_a)}{\partial\beta_a}\delta\beta\cdot\mathbf{a}_0 + \mathbf{U}_a(\beta_a)\cdot\mathbf{a}_0 + \mathbf{V}_a(\beta_b)\cdot\mathbf{b}_0 \quad (13)$$

$$\mathbf{D}_b(\beta_b)\cdot\delta\mathbf{b} = -\frac{\partial\mathbf{D}_b(\beta_b)}{\partial\beta_b}\delta\beta\cdot\mathbf{b}_0 + \mathbf{U}_b(\beta_b)\cdot\mathbf{b}_0 + \mathbf{V}_b(\beta_a)\cdot\mathbf{a}_0 \quad (14)$$

where

$$\mathbf{U}_a(\beta_a) = \mathbf{\Lambda}_a(\beta_a)\overline{\overline{\mathbf{R}}}_U(\beta_a)\mathbf{\Lambda}_a(\beta_a)[\overline{\overline{\mathbf{R}}}_B(\beta_a)-\overline{\overline{\mathbf{R}}}_B^\infty(\beta_a)] \quad (15)$$

$$\mathbf{U}_b(\beta_b) = \mathbf{\Lambda}_b(\beta_b)\overline{\overline{\mathbf{R}}}_L(\beta_b)\mathbf{\Lambda}_b(\beta_b)[\overline{\overline{\mathbf{R}}}_B(\beta_b)-\overline{\overline{\mathbf{R}}}_B^\infty(\beta_b)] \quad (16)$$

$$\mathbf{V}_a(\beta_b) = \mathbf{\Lambda}_a(\beta_b)\overline{\overline{\mathbf{R}}}_U(\beta_b)\mathbf{\Lambda}_a(\beta_b)\overline{\overline{\mathbf{T}}}_B(\beta_b) \quad (17)$$

$$\mathbf{V}_b(\beta_a) = \mathbf{\Lambda}_b(\beta_a)\overline{\overline{\mathbf{R}}}_L(\beta_a)\mathbf{\Lambda}_b(\beta_a)\overline{\overline{\mathbf{T}}}_B(\beta_a)\ . \quad (18)$$

4. Coupled-Mode Equations

In order to obtain the coupled-mode equations in a standard form, we assume that $\delta\beta \ll 2\pi / h_z$ and rewrite the amplitudes vectors as follows:

$$\mathbf{a}_0 = A(z)\mathrm{e}^{i\beta_a z}\mathbf{f}_a, \quad \mathbf{b}_0 = B(z)\mathrm{e}^{i\beta_b z}\mathbf{f}_b \quad (19)$$

where $A(z)$ and $B(z)$ denote the slowly-varying mode amplitudes which characterize the small perturbation $\delta\beta$ in the mode propagation constant, and $\mathbf{f}_a$ and $\mathbf{f}_b$ are the normalized mode eigenvectors for the waveguides "*a*" and "*b*" in isolation. Using Eq.(19), Eqs.(13) and (14) are rewritten in the following form:

$$\mathbf{D}_a(\beta_a)\cdot\delta\mathbf{a} = -\mathrm{e}^{i\beta_a z}\frac{\partial\mathbf{D}_a(\beta_a)}{\partial\beta_a}\cdot\mathbf{f}_a\left(-i\frac{d}{dz}\right)A(z) + \mathrm{e}^{i\beta_a z}\,\mathbf{U}_a(\beta_a)\cdot\mathbf{f}_a A(z) + \mathrm{e}^{i\beta_b z}\,\mathbf{V}_a(\beta_b)\cdot\mathbf{f}_b B(z) \quad (20)$$

$$\mathbf{D}_b(\beta_b)\cdot\delta\mathbf{b} = -\mathrm{e}^{i\beta_b z}\frac{\partial\mathbf{D}_b(\beta_b)}{\partial\beta_b}\cdot\mathbf{f}_b\left(-i\frac{d}{dz}\right)B(z) + \mathrm{e}^{i\beta_b z}\,\mathbf{U}_b(\beta_b)\cdot\mathbf{f}_b B(z) + \mathrm{e}^{i\beta_a z}\,\mathbf{V}_b(\beta_a)\cdot\mathbf{f}_a A(z)\,. \quad (21)$$

Equations (20) and (21) to be solved for the perturbed amplitude vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$ are singular, because $\det[\mathbf{D}_a(\beta_a)]=0$ and $\det[\mathbf{D}_b(\beta_b)]=0$. The solutions to the first-order equations are allowed only when a solvability condition is satisfied [6]. After straightforward manipulations, the solvability condition leads to the coupled-mode equations for $A(z)$ and $B(z)$ as follows:

$$\frac{d}{dz}A(z) = i\kappa_{aa}A(z) + i\,\mathrm{e}^{-i\Delta\beta z}\,\kappa_{ab}B(z) \quad (22)$$

$$\frac{d}{dz}B(z) = i\kappa_{bb}B(z) + i\,\mathrm{e}^{i\Delta\beta z}\,\kappa_{ba}A(z) \quad (23)$$

with

$$\kappa_{aa} = \frac{\mathbf{g}_a\cdot\mathbf{U}_a(\beta_a)\cdot\mathbf{f}_a}{\mathbf{g}_a\cdot\dfrac{\partial\mathbf{D}_a(\beta_a)}{\partial\beta_a}\cdot\mathbf{f}_a}, \quad \kappa_{ab} = \frac{\mathbf{g}_a\cdot\mathbf{V}_a(\beta_b)\cdot\mathbf{f}_b}{\mathbf{g}_a\cdot\dfrac{\partial\mathbf{D}_a(\beta_a)}{\partial\beta_a}\cdot\mathbf{f}_a} \quad (24)$$

$$\kappa_{bb} = \frac{\mathbf{g}_b\cdot\mathbf{U}_b(\beta_b)\cdot\mathbf{f}_b}{\mathbf{g}_b\cdot\dfrac{\partial\mathbf{D}_b(\beta_b)}{\partial\beta_b}\cdot\mathbf{f}_b}, \quad \kappa_{ba} = \frac{\mathbf{g}_b\cdot\mathbf{V}_b(\beta_a)\cdot\mathbf{f}_a}{\mathbf{g}_b\cdot\dfrac{\partial\mathbf{D}_b(\beta_b)}{\partial\beta_b}\cdot\mathbf{f}_b} \quad (25)$$

$$\Delta\beta = \beta_a(\omega) - \beta_b(\omega) \quad (26)$$

where $\mathbf{g}_a$ and $\mathbf{g}_b$ are the right eigenvectors that satisfy $\mathbf{D}_a^{\mathrm{T}}(\beta_a)\cdot\mathbf{g}_a = \mathbf{0}$ and $\mathbf{D}_b^{\mathrm{T}}(\beta_b)\cdot\mathbf{g}_b = \mathbf{0}$, respectively. The solution to Eqs.(22) and (23) describes the power transfer characteristics between two-parallel photonic crystal waveguides "*a*" and "*b*" coupled through the photonic crystal barrier of a finite thickness. The perturbed amplitude vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$ can be determined from Eqs.(20) and (21) by using the same procedure as reported in [6].

5. Conclusions

The coupled-mode equations for coupled two-dimensional photonic crystal waveguides have been derived using a perturbation approach. The numerical calculation of the self- and mutual-coupling coefficients (24) and (25) is under consideration.

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