

# Non-iterative electromagnetic imaging of perfectly conducting screens from limited range far-field data

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## Abstract

We propose a non-iterative multi-frequency algorithm for the imaging of arbitrarily shaped perfectly conducting screens from limited range far-field data. Based on the factorization of the Multi-Static Response (MSR) matrix collected in the far-field at multiple frequencies in Transverse Magnetic (TM) mode, the imaging problem can be solved through the simple analysis of corresponding eigenvectors. Numerical examples illustrate how the proposed algorithm behaves.

**Keywords :** Perfectly conducting cracks Multi-Static Response matrix Numerical examples

## 1. Introduction

The problem of imaging the shape of unknown perfectly conducting screens from the knowledge of the incident and measured far-field patterns of the scattered field is an old but interesting and important problem in electromagnetic and acoustics which is arising in physics, medical science, material engineering, and so on. Over a number of decades, based on iterative schemes, various imaging methods have been developed, refer to [1,4,9].

However, in order to obtain a good imaging result via these methods, one needs a good initial guess which is close enough to the unknown crack. Without it, one faces large computational costs with the risk of non-convergence. Moreover, such schemes often require regularization terms highly depending on the problem at hand. In order to avoid these difficulties, many authors suggested alternative non-iterative imaging methods, which at least provide good initial guesses. Among them, MUSIC (MUltiple SIgnal Classification)-type algorithm is successfully applied for imaging not only penetrable thin inclusions [8] but also perfectly conducting screens [2,3,7]. However, under the limited range incident and observation directions that can be enforced in practical situations, e.g., detection of obstacles embedded in the ground, one cannot successfully obtain imaging results [6]. In order to improve on this limitation, an effective non-iterative imaging algorithm with limited range data for imaging perfectly conducting screens is proposed herein. This is based on a factorization of the Multi-Static Response (MSR) matrix collected from far-field pattern of the scattered field at multiple frequencies in the Transverse Magnetic (TM) mode.

We will organize this paper as follows. In section 2, the two-dimensional direct scattering problem is briefly discussed. In section 3, a multi-frequency based non-iterative imaging algorithm is sketched. In section 4, various numerical results are proposed in order to show the effectiveness of the proposed algorithm. The short conclusion follows in section 5.

## 2. Direct scattering problem: Helmholtz equation with Dirichlet boundary condition

We consider the two-dimensional direct scattering by a perfectly conducting screen. The screen is a smooth, non-intersecting arc which can be represented as

$$\Gamma = \{z(s) : s \in [-1, 1]\}$$

where  $z(s)$  is an injective piecewise function.

Let  $u(x)$  denote the time-harmonic electromagnetic total field which satisfies the following two-dimensional Helmholtz equation with Dirichlet boundary condition (Transverse Magnetic – TM – polarization)

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0, & x \in \mathbb{R}^2 \setminus \Gamma \\ u(x) = 0, & x \in \Gamma \end{cases} \quad (1)$$

where  $k$  denotes the wave number. Let us notice that the total field can be set as  $u(x) = u_0(x) + u_s(x)$ ,  $u_0(x) = \exp(ik\theta \cdot x)$  the given incident field for incident direction  $\theta \in S^1$  (two-dimensional unit circle) and  $u_s(x)$  the unknown scattered field which satisfies the Sommerfeld radiation condition. The latter can be represented as a single-layer potential with unknown density function  $\varphi$

$$u_s(x) = \int_{\Gamma} G(x, y) \varphi(y; \theta) dy, \quad x \in \mathbb{R}^2 \setminus \Gamma, \quad (2)$$

where  $G$  is the two-dimensional Green function or fundamental solution to the Helmholtz equation

$$G(x, y) = \frac{i}{4} H_0^1(k|x - y|), \quad x \neq y$$

The far-field pattern  $u_\infty$  of the scattered field  $u_s$  is defined on  $S^1$  such that

$$u_s(x) = \frac{\exp(ik|x|)}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}; \theta) + O\left(\frac{1}{|x|}\right) \right\}$$

uniformly in all directions  $x/|x|$  and  $|x| \rightarrow \infty$ . From the above representation and the asymptotic formula for the Green function, the far field pattern is given by

$$u_\infty(\hat{x}; \theta) = \frac{1+i}{4\sqrt{\pi k}} \int_{\Gamma} e^{-ik\hat{x} \cdot y} \varphi(y; \theta) dy, \quad \hat{x} \in S^1 \quad (3)$$

### 3. Multi-frequency based non-iterative imaging algorithm

In this section, we apply the far-field pattern formula (3) in order to design an imaging algorithm. For that purpose, we use the eigenvalue structure of the Multi-Static Response (MSR) matrix. If the transmitters and receivers coincide, the MSR matrix  $K$  can be written as

$$K = \int_{\Gamma} E(y) F(y)^T dy \quad (4)$$

where

$$\begin{aligned} E(y) &= (\exp(-ik\hat{x}_1 \cdot y), \exp(-ik\hat{x}_2 \cdot y), \dots, \exp(-ik\hat{x}_N \cdot y))|_{\hat{x}_j = -\theta_j} \\ &= (\exp(ik\theta_1 \cdot y), \exp(ik\theta_2 \cdot y), \dots, \exp(ik\theta_N \cdot y)) \end{aligned}$$

and

$$F(y) = (\varphi(y; \theta_1), \varphi(y; \theta_{12}), \dots, \varphi(y; \theta_N))^T$$

Formula (4) is a factorization of the MSR matrix that separates the known incoming plane wave information from the unknown information. The range of  $K$  is determined by the span of the  $E$  corresponding to  $\Gamma$ , i.e., we can define a signal subspace by using a set of left singular vectors of  $K$ , refer to [5].

Let us assume that the crack is divided into  $M$  different segments of size of order half the wavelength. Having in mind the Rayleigh resolution limit, only one point, say  $y_m$  for  $m = 1, 2, \dots, M$ , at each segment is expected to contribute to the image space of the response matrix  $K$ , refer to [2, 3, 5, 6, 7, 8]. Since transmitters and receivers coincide, the MSR matrix  $K$  is complex symmetric, therefore  $K$  can be written

$$K = USU^T \approx \sum_{m=1}^M u_m(k) s_m u_m^T(k). \quad (5)$$

Define a vector

$$d(y; k) = (\exp(ik\theta_1 \cdot y), \exp(ik\theta_2 \cdot y), \dots, \exp(ik\theta_N \cdot y))^T \quad \text{and} \quad \hat{d}(y; k) = \frac{d(y; k)}{\|d(y; k)\|}$$

Then, in accord with [2, 5], for some values  $\sigma_m$ ,  $m = 1, 2, \dots, M$ , we can recognize that

$$u_m(k) \sim \exp(i\sigma_m) \hat{d}(y_m; k).$$

Since the first  $M$  columns of the matrix  $U$ ,  $\{u_1(k), u_2(k), \dots, u_M(k)\}$ , are orthonormal, one can easily find that

$$\hat{d}(y; k)^* u_m(k) \neq 0, \quad y = y_m \quad \text{and} \quad \hat{d}(y; k)^* u_m(k) \approx 0, \quad y \neq y_m, \quad (6)$$

where  $*$  is the mark of complex conjugate. With this, we construct the following normalized image function with MSR matrices at multiple frequencies  $\{k_f: f = 1, 2, \dots, F\}$  as

$$W(y) = \frac{1}{F} \sum_{f=1}^F \sum_{m=1}^{M_f} |\hat{d}(y; k_f)^* u_m(k_f)|^2, \quad (7)$$

where  $M_f$  is number of nonzero singular values of MSR matrix at  $k_f$ . Then, from observation (6),  $W(y)$  will exhibit peaks of magnitude of (close to) 1 at location  $y_m$  for  $m = 1, 2, \dots, M$  and of small magnitude at others. However, since the matrix  $K$  is not Hermitian, a Singular Value Decomposition (SVD) must be applied as follows:

$$K = USV^* \approx \sum_{m=1}^M u_m(k) s_m v_m^*(k) = \sum_{m=1}^M u_m(k) s_m \bar{v}_m^T(k)$$

and we will use following image function instead of (7)

$$W(y) = \frac{1}{F} \sum_{f=1}^F \sum_{m=1}^{M_f} |\hat{d}(y; k_f)^* u_m(k_f)| |\hat{d}(y; k_f)^* \bar{v}_m(k_f)|.$$

A suitable number of  $M_f$  for each frequency can be found via careful thresholding, see [7,8] for instance. Let us notice that based on the statistical hypothesis testing, multiple frequencies are expected to enhance the imaging performance, refer to [2,5,6].

## 4. Numerical experiments

We present some numerical examples for imaging arc-like screens. Throughout this section, the applied wave number is taken of the form  $k_f = 2\pi/\lambda_f$ ; here  $\lambda_f$ ,  $f = 1, 2, \dots, 10$ , is the given wavelength. In this paper, wave numbers  $k_f$  are always equi-distributed in the interval  $[k_1, k_{10}]$ .

Two  $\Gamma_j$  are chosen for illustration:

$$\Gamma_1 = \{(z - 0.2, -0.5z^2 + 0.6) : z \in [-0.5, 0.5]\},$$

$$\Gamma_2 = \{(z + 0.2, z^3 + z^2 - 0.6) : z \in [-0.5, 0.5]\}.$$

The search domain  $\Omega$  is chosen as  $\Omega = [-1, 1] \times [-1, 1]$  and the observation directions are taken as

$$\hat{y}_j = -(\cos \zeta_j, \sin \zeta_j), \quad \zeta_j = \alpha + (\beta - \alpha) \frac{j-1}{N-1} \quad \text{for} \quad \alpha = \frac{\pi}{6} \quad \text{and} \quad \beta = \frac{5\pi}{6} \quad (8)$$

for  $j = 1, 2, \dots, N$ . The incident directions are opposite to the observation ones. After obtaining the scattered field data, a white Gaussian noise with 20dB Signal-to-Noise Ratio (SNR) is added to the unperturbed data for showing the robustness of the algorithm. In order to obtain the number of nonzero singular values  $M_f$  for each frequency, a 0.01-threshold scheme (choosing first  $j$  singular values  $s_j$  satisfying  $s_j/s_1 \geq 0.01$ ) is adopted.

Let us first work with  $\Gamma_1$ ,  $K$  being collected for  $N = 20$ . In this case, the map of  $W(y)$  offers a good result so that one can successfully recognize  $\Gamma_1$ . Now, let us consider a crack of complex shape  $\Gamma_2$  with MSR matrix  $K$  for  $N = 20$ . Although some part of  $\Gamma_2$  is not retrieved, the proposed algorithm gives a good result.

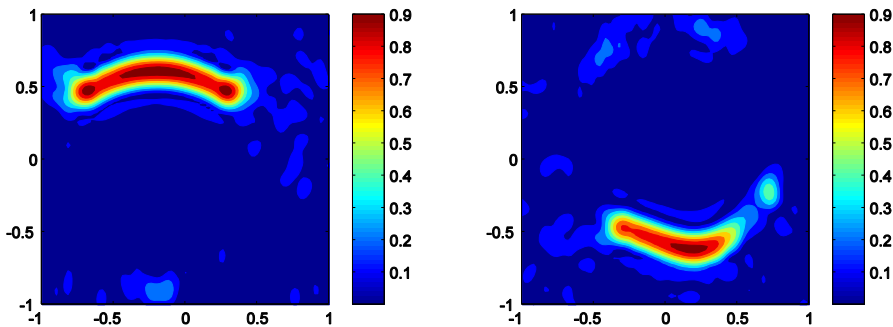


Figure 1: Map of  $W(y)$  for  $\Gamma_1$  (left) and  $\Gamma_2$  (right).

It could be applied directly to multiple well-separated screens. For that purpose, let us consider  $\Gamma_1 \cup \Gamma_2$ . As we see in Figure 2, it is hard to recognize the shape of  $\Gamma_2$  due to the range of directions of observation. This result shows the limitation of the algorithm. If one wants to retrieve  $\Gamma_2$ , different directions of observation are needed.

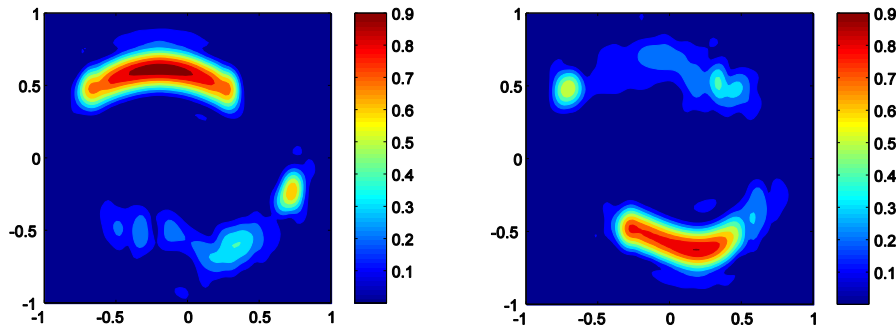


Figure 2: Map of  $W(y)$  for  $\Gamma_1 \cup \Gamma_2$  when  $\alpha = \pi/6, \beta = 5\pi/6$  (left) and  $\alpha = -\pi/6, \beta = \pi/6$  (right).

## 5. Concluding remarks

In this paper, a multi-frequency non-iterative algorithm has been proposed to image perfectly conducting, arc-like cracks modeled via a Dirichlet boundary condition (TM polarization) based on a factorization of the collected Multi-Static Response (MSR) matrix.

Throughout various numerical simulations, it has been shown that the proposed algorithm is effective and robust with respect to the random noise. Moreover, it can be easily applied to multiple screens. Though we have considered a TM polarization case only, the proposed method can be carried out onto the study in the Transverse Electric (TE) polarization, and it will be a forthcoming work.

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