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A Fixed Point Theorem for Successively Recurrent System of Set-Valued Mapping Equations and its Applications

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Abstract—Let us introduce n ($\geq$ 2) mappings $f_i (i = 1, 2, \cdots, n \equiv 0)$ defined on Banach spaces X_{i-1} ($i = 1, 2, \cdots, n \equiv 0$), respectively, and let $f_i : X_{i-1} \to Y_i$ be completely continuous on bounded convex closed subsets $X^{(0)}_{i-1} \subset X_{i-1}$. Moreover, let us introduce n set-valued mappings $F_i : X_{i-1} \times Y_i \to \mathcal{F}(X_i)$(the family of all non-empty compact subsets of X_i), ($i = 1, 2, \cdots, n \equiv 0$). Here, we have a fixed point theorem on the successively recurrent system of set-valued mapping equations:$x_i \in F_i(x_{i-1}, f_i(x_{i-1}))$, ($i = 1, 2, \cdots, n \equiv 0$). This theorem can be applied immediately to analysis of the availability of system of circular networks of channels undergone by uncertain fluctuations and to evaluation of the tolerability of behaviors of those systems. In this paper, mathematical situation and detailed proof in weak topology are discussed, about this theorem.

1. Introduction

In general, by the set-valued mapping F defined on a Banach space X is meant a correspondence in which a set $F(x)$ in a space Y is specified in correspondence to any point x in X.In particular, when $F(X) \subset X$, and if there exists a point x^* such that $x^* \in F(x^*)$, x^* is called a fixed point of F [1]. The author gave some types of discussions of uncertain fluctuation problems of nonlinear mapping equations, say $y = f(x)$, where f is a completely continuous on X, giving a composition type of fixed point theorem: $x^* \in F(x^*, f(x^*))$ [2, 3]. This type of fixed point theorem has been applied by the author himself vigorously to many important problems as mathematical foundations of sensitivity analysis, modeling and simulation synthesis, security analysis, tolerable communication system analysis and design, and so on [1]. Thereafter, the author gave more refined type of fixed point theorems for the system of set-valued mapping equations, in order to treat with more complex systems [4]-[9].

In complex large-scale so called as multi-media network systems, the successively recurrent circular connectin of channels often plays an important role as a typical element network of local area networks(LAN). However, in large-scale circular networks, whenever some undesirable deviations are induced into their element channels, even if their influences are originally small to every element channel itself, total affections may be successively accumulated step by step, and as results, may grow so serious that the network itself becomes useless. Therefore, we must carefully evaluate and control those deviations such that overall behaviors of respective channel outputs can be led, in real time, into "available" or "tolerable" regions assigned in advance.

In such a reason, the author thereafter presented a fixed point theorem for a successively recurrent system of set-valued mapping equations, with its detailed proof, under a natural assumption, and refered to available behaviors of signals to be appeared in every portion of succesively recurrent circular channels disturbed by undesirable uncertainties [10]. This paper is its refined theory of the same problem under more wide conditions, through a precise deduction in weak topology.

2. Fixed Point Theorem for Successively Recurrent-System of Set-Valued Mapping Equations

Here, we will present a refined theory of the fixed point theorem for the succesively recurrent system of set-valued mapping equations, with the proof in weak topology.

For the first step, let us introduce reflexive real Banach spaces X_i ($i = 1, \ldots, n \equiv 0$), in which the norm is represented by $\| \cdot \|$, and also, let us define there non-empty bounded closed convex subsets $X^{(0)}_i$ ($i = 1, \ldots, n \equiv 0$). Let X'_i be the dual space of X_i and let us introduce a weak topology $\sigma\left(X_i, X'_i\right)$ into X_i. Then, X_i is locally convex topological linear space,and therefore, $X^{(0)}_i$ is weakly closed and weakly compact. Further, let us consider another real Banach space Y_i ($j = 1, \ldots, n \equiv 0$) in which the norm is represented by $\| \cdot \|$.

Now, let us introduce n (≥ 2) mappings f_i ($i = 1, \ldots, n \equiv 0$) defined on X_{i-1} ($i = 1, \ldots, n \equiv 0$), respectively, and let $f_i : X_{i-1} \to Y_i$ be completely continuous on bounded convex closed subset $X^{(0)}_{i-1} \subset X_{i-1}$ ($i = 1, \ldots, n \equiv 0$) (Figure 1).

Moreover, let us introduce n set-valued mappings $F_i : X_{i-1} \times Y_i \to F(X_i)$ (the family of all non-empty compact subsets of X_i) ($i = 1, \ldots, n \equiv 0$).

Here, we can recognize that for any $x_{i-1} \in X^{(0)}_{i-1}$ and $f_i(x_{i-1}) \in Y_i$, we have $F^{(0)}_i(x_{i-1}; f_i(x_{i-1})) \triangleq F^{(0)}_i(x_{i-1}; f_i(x_{i-1})) \cap X^{(0)}_i \neq \phi$, and moreover, there exist projection points $\tilde{x}'_i$ of arbitry point $x'_i \in X^{(0)}_i$ upon the set

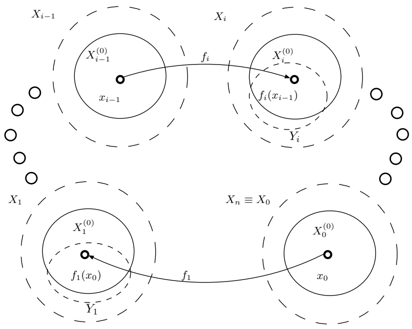

Figure 1: Successively recurrent connection of Banach spaces X_i and mappings f_i $(i = 1, \cdots, n \equiv 0)$.

$F_i^{(0)}(x_{i-1}; f_i(x_{i-1}))$ such that

$$\left\| \tilde{x}_i{}' - x_i' \right\| = \min\{\left\| x_i' - z_i \right\| \mid z_i \in F_i^{(0)}(x_{i-1}; f_i(x_{i-1}))\}. \tag{1}$$

Now, let us introduce a series of assumptions:

Assumption 1 *Let the mapping* $f_i : X_{i-1}^{(0)} \to f_i(X_{i-1}^{(0)}) \subset Y_i$ *be completely continuous in the sense that when a weakly convergent net* $\{x_i^\nu\}$ $(\nu \in J$ *: a directive set) weakly converges to* $\bar{x}_i$*, then the sequence* $\{f_i(x_{i-1}^\nu)\}$ *has a subsequence which strongly converges to* $f_i(\bar{x}_{i-1})$ *in* Y_i*.*

Assumption 2 *Let the set-valued mapping* $F_i : X_{i-1}^{(0)} \times Y_i \to \mathcal{F}(X_i)$ *(a family of all non-empty compact subsets of* X_i*) satisfies the following Lipschits condition wth respect to the Hausdorff distance* d_H*, that is, there are two kinds of constants* $a_i > 0$ *and* $b_i > 0$ *such that for any* $x_{i-1}^{(1)}, x_{i-1}^{(2)} \in X_{i-1}$ *and for any* $y_i^{(1)}, y_i^{(2)} \in Y_i$*,* F_i *satisfied the inequality*

$$d_H\left(F_i(x_{i-1}^{(1)}; y_i^{(1)}), F_i(x_{i-1}^{(2)}; y_i^{(2)})\right) \leq a_i \cdot \left\| x_{i-1}^{(1)} - x_{i-1}^{(2)} \right\| + b_i \cdot \left\| y_i^{(1)} - y_i^{(2)} \right\|, \tag{2}$$

where Lipschitz constants a_i $(i = 1, \ldots, n \equiv 0)$ *are confined by*

$$0 < a_1 \cdot a_2 \cdot \ldots \cdot a_n < 1. \tag{3}$$

Here, the Hausdorff distance d_H *between two sets* S_1 *and* S_2 *is defined by*

$$d_H(S_1, S_2) \triangleq \max\{\sup\{d(x_1, S_2) | x_1 \in S_1\}, \sup\{d(x_2, S_1) | x_2 \in S_2\}\}, \tag{4}$$

where $d(x, S) \triangleq \inf\{\|x - z\| \mid z \in S\}$ *is the distance between a point* x *and a set* S*.*

Then, under these preparations, we have an important lemma on the system of set-valued mapping equations:

$$x_i \in F_i(x_{i-1}; f_i(x_{i-1})), \ (i = 1, \ldots, n \equiv 0). \tag{5}$$

Lemma 1 *For all* i $(i = 1, \ldots, n \equiv 0)$*, let us adopt arbitrary points* $x_i^{(0)} \in X_i^{(0)}$ *and also fix all values of* $f_i(x_{i-1}^{(0)})$ $(i = 1, \ldots, n \equiv 0)$*. Now, for every* i*, let us consider a sequence* $\{x_i^{(\nu)}\}(\nu = 0, 1, 2, \ldots)$ *starting from the above adopted point* $x_i^{(0)}$*, and with each* $x_i^{(\nu)} \in X_i^{(0)}$ *as the projection point of* $x_i^{(\nu-1)} \in X_i^{(0)}$ *upon the set* $F_i(x_{i-1}^{(\nu-1)}; f_i(x_{i-1}^{(0)}))$*, but for any number* $m \geq 1$ $(m \leq n - i)$*,* $x_{i+m}^{(\nu)} \in X_{i+m}^{(0)}$ *is specified as the projection point from* $x_{i+m}^{(\nu-1)} \in X_{i+m}^{(0)}$ *upon the set* $F_i(x_{i+m-1}^{(\nu)}; f_{i+m}(x_{i+m-1}^{(0)}))$*. Then, these sequences* $\{x_i^{(\nu)}\}$ $(i = 1, \ldots n \equiv 0; \nu = 0, 1, 2, \ldots)$ *are Cauchy sequences, having their own limit points* $\bar{x}_i \in X_i^{(0)}$*, respectively, such that*

$$\bar{x}_i \in F_i^{(0)}(\bar{x}_{i-1}; f_i(x_{i-1}^{(0)})), \ (i = 1, \ldots, n \equiv 0). \tag{6}$$

All limit points $\bar{x}_i$ $(i = 1, \ldots, n \equiv 0)$ depend on their starting points $x_i^{(0)}$ and parameters $y_i^{(0)} \triangleq f_i(x_{i-1}^{(0)})$, respectively. These correspondences may be multi-valued, in general, and hence, can be represented by set-valued continuous mappings defined on each domain:

$$\bar{x}_i \in H_i(x_i^{(0)}, y_i^{(0)}), \ (i = 1, \ldots, n \equiv 0). \tag{7}$$

If these mappings have fixed points x_i^* in respective domain, or in respective bounded convex closed subsets $X_i^{(0)}$: i.e.,

$$x_i^* \in H_i(x_i^*; y_i^*) \in X_i^{(0)}, \ (i = 1, \ldots, n \equiv 0), \tag{8}$$

where $y_i^* \triangleq f_i(x_{i-1}^*)$, these relations imply that

$$x_i^* \in F_i^{(0)}(x_{i-1}^*; f_i(x_{i-1}^*)), \ (i = 1, \ldots, n \equiv 0). \tag{9}$$

This result means that the solution set $\{x_i^*\}$ $(i = 1, \ldots, n \equiv 0)$ of the system of set-valued nonlinear mapping equations (5):

$$x_i \in F_i(x_{i-1}; f_i(x_{i-1})), \ (i = 1, \ldots, n \equiv 0) \tag{10}$$

can be obtained in connection with the set of limit points $\{\bar{x}_i\}$ $(i = 1, \ldots, n \equiv 0)$ of Cauchy sequences $\{x_i^{(\nu)}\}$ $(i = 1, \ldots, n \equiv 0; \nu = 0, 1, 2, \ldots)$.

Here, in order to verify the existance of the fixed point x_i^* of H_i, from the stand point refined in the weak topology, now, it is very convenient to apply the well-known fixed point theorem for set-valued mapping:

Lemma 2 (Ky Fan [11]) *Let* X_i *be a locally convex topological linear space, and* $X_i^{(0)}$ *be a non-empty convex compact subset of* X_i*. Let* $\mathcal{H}_c(X_i^{(0)})$ *be the family of all non-empty closed convex subsets of* $X_i^{(0)}$*. Then, for upper semi-continuous set-valued mapping* $H_i : X_i^{(0)} \to \mathcal{H}_c(X_i^{(0)})$*, there exists a fixed point* $x_i^* \in X_i^{(0)}$ *such that* $x_i^* \in H_i(x_i^*)$*.*

In order to apply this lemma to our problem, we must verify that the above-defined set-valued mapping $H_i(x_i) \equiv$

$H_i(x_i; y_i^{(0)})$ is upper semicontinuous, and its range is closed and convex.

In the first place, the closedness of the range of $H_i(x_i; y_i^{(0)})$ is easily verified from Assumption 2. For the verification of the convexity, it is sufficient to add the following assumption:

Assumption 3 (Rockafellar [12]) *For any* $x_{i-1}^{(1)}, x_{i-1}^{(2)} \in X_{i-1}^{(0)}$*, and for any constant* r $(0 < r < 1)$*, uniformly with respect to every* $y_i \in Y_i$*,* F_i *satisfies the relation:*

$$\begin{aligned} & r \cdot F_i(x_{i-1}^{(1)}; y_i) + (1-r) \cdot F_i(x_{i-1}^{(2)}; y_i) \\ & \subset F_i(r \cdot x_{i-1}^{(1)} + (1-r) \cdot x_{i-1}^{(2)}; y_i). \end{aligned} \tag{11}$$

In fact, under Assumption 3, we have

$$\begin{aligned} & r x_i^{(1)} + (1-r) x_i^{(2)} \\ & \in r \cdot F_i(x_{i-1}^{(1)}; y_i) + (1-r) \cdot F_i(x_{i-1}^{(2)}; y_i) \\ & \subset F_i(r \cdot x_{i-1}^{(1)} + (1-r) \cdot x_{i-1}^{(2)}; y_i), \end{aligned} \tag{12}$$

for any $x_i^{(\nu)} \subset F_i(x_{i-1}^{(\nu)}; y_i)$ $(\nu = 1, 2)$: i.e., for any $x_i^{(\nu)} \in H_i(x_i^{(\nu)}, y_i)$ $(\nu = 1, 2)$. This relation means the convexity of $H_i(x_i; y_i)$.

Lastly, in order to verify the upper semicontinuity, we should prove that if an arbitrary weakly convergent net $\{x_i^\nu\}$ $(\nu \in J)$ in $X_i^{(0)}$ weakly converges to $\bar{x}_i$ and if the weakly convergent net $\{z_i^\nu\}$ $(\nu \in J)$ in $X_i^{(0)}$ made from $z_i^\nu \in H_i(x_i^\nu)$ weakly converges to $\bar{z}_i$, we have $\bar{z}_i \in H_i(\bar{x}_i)$.

For this purpose, we can use the following lemma:

Lemma 3 (Nadler[13]) *Let* X_i *be a Banach space, and let* G_i^ν $(\nu \in J)$ *and* $\bar{G}_i : X_i \to \mathcal{F}_c(X_i)$ *(the family of all non-empty compact subsets of* X_i*) be set-valued mappings contracting with respect to the Hausdorff distance* d_H *: e.g., there exists a constant* a_i $(0 < a_i < 1)$ *such that for any* $z_i^{(1)}, z_i^{(2)} \in X_i$*,* G_i *satisfies the inequality*

$$d_H\left(G_i^\nu(z_i^{(1)}), G_i^\nu(z_i^{(2)})\right) \le a_i \left\| z_i^{(1)} - z_i^{(2)} \right\|. \tag{13}$$

Now, let $\{G_i^\nu\}$ be uniformly convergent to $\bar{G}_i$ in the distance d_H. Let z_i^ν be fixed point of G_i^ν. Then, we can find that the sequence $\{z_i^\nu\}$ $(\nu \in J)$ has a convergent subsequence $\{z_i^{\nu m}\}$ and its limit point $\bar{z}_i$ is a fixed point of $\bar{G}_i : \bar{z}_i \in \bar{G}_i(\bar{z}_i)$.

From Assumprion 1, we remember that when any weakly convergent net $\{x_i^\nu\}$ $(\nu \in J)$ weakly converges to $\bar{x}_i$, the net $\{f_i(x_{i-1}^\nu)\}$ has a subsequence $\{f_i(x_{i-1}^{\nu m})\}$ strongly convergent to $f_i(\bar{x}_{i-1})$. On the other hand, from Assumption 2, we have

$$\begin{aligned} & \sup_{x_{i-1} \in X_{i-1}} d_H\left(F_i(x_{i-1}; f_i(x_{i-1}^{\nu m})), F_i(x_{i-1}; f_i(\bar{x}_{i-1}))\right) \\ & \qquad \le b_i \cdot \left\| f_i(x_{i-1}^{\nu m}) - f_i(\bar{x}_{i-1}) \right\| \to 0. \end{aligned} \tag{14}$$

This implies that the sequence of set-valued mappings $\{F_i^{(0)}(x_{i-1}; f_i(x_{i-1}^{\nu m}))\}$ uniformly converges to $F_i^{(0)}(x_{i-1}; f_i(\bar{x}_{i-1}))$, in the distance d_H. Thus, from this deduction, substituting $G_i^\nu, G_i^{\nu m}$ and $\bar{G}_i$ of Lemma 3, by $F_i^{(0)}(z_{i-1}; f_i(z_{i-1}^\nu)), F_i^{(0)}(z_{i-1}; f_i(z_{i-1}^{\nu m}))$ and $F_i^{(0)}(z_{i-1}; f_i(\bar{z}_{i-1}))$, respectively, we can apply Lemma 3, and hence, we find that the sequence of fixed points $\{z_i^{\nu m}\} : z_i^{\nu m} \in F_i^{(0)}(z_{i-1}^{\nu m}; f_i(z_{i-1}^{\nu m}))$, i.e., $z_i^{\nu m} \in H_i(z_i^{\nu m})$, strongly, and therefore, weakly converges to the fixed point

$$\bar{z}_i : \bar{z}_i \in F_i^{(0)}(\bar{z}_{i-1}; f_i(\bar{z}_{i-1})),$$

i.e., $\bar{z}_i \in H_i(\bar{z}_i)$.

As a result, we have the theorem:

Theorem 1 *Let* X_i *be a reflexive real Banach space, and* $X_i^{(0)}$ *be a non-empty bounded closed convex subset of* X_i*. By the dual space* X_i'*, let us introduce a weak topology* $\sigma(X_i, X_i')$ *into* X_i*. Let* f_i *and* F_i *be deterministic and set-valued mappings, respectively, which satisfy the series of Assumptions 1 to 3. Then, we have a Cauchy sequence* $\{x_i^\nu\} \subset X_i^{(0)}$ $(\nu = 0, 1, 2, \ldots)$*, introduced by the successive procedure in Lemma 1. This sequence has a set of limit points* $\{\bar{x}_i\}$*, and we can define a set-valued mapping* H_i *by the correspondence from the arbitrary starting point* $z_i^{(0)} \equiv x_i \in X_i^{(0)}$ *to the set of limit points* $\{\bar{x}_i\}$ *in* $X_i^{(0)} : \bar{x}_i \in H_i(x_i)$*. This set-valued mapping* H_i *has a fixed point* x_i^* *in* $X_i^{(0)}$*, which is, in turn, the solution of the system of set-valued mapping equations (10).*

3. An Application to Systems of Successively Recurrent Circular Channels with Uncertainties

The fixed point theorem above-derived can be applied immediately to analysis of the availability of system of successively recurrent circular channels with uncertainties and to evaluation of the tolerability of behaviors of those systems. Let us consider a successively recurrent circular system of communication channels which consists of n stations and n unilateral channels (Figure 2). The station (i) is operated by operator (i), $(i = 1, 2, \cdots, n \equiv 0)$. Thus, in consideration of the output signal x_{i-1} of the channel $(i-1)$, the operator $(i-1)$, at the station $(i-1)$, operates his own input signal u_{i-1}^0 and gives the input signal w_{i-1} into the channel (i). The channel (i) transfers this input signal w_{i-1}

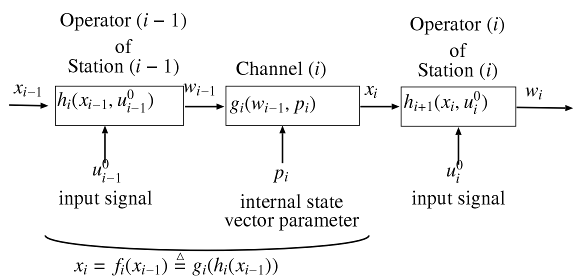

Figure 2: The successively recurrent circular chain of stations (i) and channels (i) $(i = 1, 2, \cdots, n \equiv 0)$

to the output side as an input signal x_i of the station (i). The function of the operator ($i-1$) is described as

$$w_{i-1} = h_i(x_{i-1}, u^0_{i-1}), \tag{15}$$

and the function of the channel (i) is described as

$$x_i = g_i(w_{i-1}, p_i), \tag{16}$$

where p_i denotes the internal state vector parameter of channel (i), representative of whole internal structures and parameters. For fixed u^0_{i-1} and p_i, functions h_i and g_i can be abbreviated as $h_i(x_{i-1})$ and $g_i(w_{i-1})$, respectively. Incidentally, we denote

$$f_i(x_{i-1}) \triangleq g_i(h_i(x_{i-1})). \tag{17}$$

When uncertainty may be induced into the internal state vector parameter p_i, then the original function $f_i(x_{i-1})$ of the channel (i) is diversified in the form of the set-valued mapping:

$$\begin{aligned} &F_i(x_{i-1}, f_i(x_{i-1})) \\ &\triangleq G_i(h_i(x_{i-1}), f_i(x_{i-1})) \\ &\equiv G_i(h_i(x_{i-1}), g_i(h_i(x_{i-1}))). \end{aligned} \tag{18}$$

Thus, the behavior of the channel (i) can be described in the form of Eq.(5), and therefore, the analysis of this type of succesively recurrent system of unilateral channels disturbed by undesirable uncertainties are successfully accomplished, by immediate application of the above-described fixed point theorem for system of set-valued mappings.

4. Concluding Remarks

The solution set $\{x^*_i\}(i = 1, 2, \cdots, n \equiv 0)$ in $\{X^{(0)}_i\}(i = 1, 2, \cdots, n \equiv 0)$ of the system of equations (10) can be recognized as available behaviors of signals to be appeared at output-terminals of unilateral channels $\{(i)\}(i = 1, 2, \cdots, n \equiv 0)$. Here, "available behavior" means that solutions x^*_i appear in each "tolerable" region $X^{(0)}_i$ assigned in advance, and behaviors represented by solutions x^*_i can be considered as available for real performances of channels undergone by undesirable uncertain fluctuations.

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