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A Heuristic Approach to Graph Coloring Problems Using a Complex-Valued Neural Network

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Abstract—Complex-valued neural networks are widely used to process not only complex-valued information but also multivalued one. For instance, multivalued representation with a complex-valued neuron is more suitable for dealing with multicolor images than binary representation with a real-valued neuron, because less neurons are needed. In seeking for another beneficial application of a complex-valued neuron, we demonstrate that a complex-valued neural network can provide a heuristic approach to graph coloring problems.

1. Introduction

The application field of complex-valued neural networks have been growing, because complex-valued representation is natural for wave phenomena with amplitude and phase information [1]. In recent years, therefore, it has been increasingly important to explore the fundamental theory of complex-valued neural networks for their developments and practical applications. The conventional real-valued neurons have often been used to represent binary information and data with a two-state activation function. Similarly, a complex-valued neuron enables to well represent multivalued information by using a multilevel activation function. In this study, we focus on a complex-valued neural network for processing multivalued information and its application.

Although the concept of a complex-valued neuron was presented in the early 1970s [2], its mathematical property is still under investigation. Some complex-valued neurons take into consideration both amplitude and phase information, while others only uses phase information. The latter type of complex-valued neuron, whose state is defined on the unit circle in the complex domain, is called a phasor model [3] or a multivalued neuron [4]. The characteristics of the complex-valued neuron, which differ from those of a real-valued neuron, are the circularity of the phase and the complex number operations. For realizing a K-valued neuron, the unit circle is equally divided into K arcs with the same angle for any integer $K \geq 2$ as explained later. The neuronal state is transformed into one of the K-valued states by an activation function. A complex-valued Hopfield network composed of such complex-valued neurons, which can be regarded as a generalization of the Hopfield network [5], has been constructed and applied to multistate associative memory [6]. The complex-valued Hopfield model can deal with multivalued states with keeping some important properties of the Hopfield network. Namely, it is possible to update neuronal states so that a well defined complex-valued energy function can be monotonically decreasing and guarantees convergence of a network state in a local minimum of the energy landscape. Gray-level image reconstruction is a good application to be processed by the complex-valued Hopfield network with some modifications [7, 8]. Another possible application is to find an optimal solution of a combinatorial optimization problem if the objective function to be minimized can be rewritten in a complex quadratic form.

Following the seminal work by Hopfield and Tank [9], binary and real-valued neural networks have been successfully applied to a variety kinds of combinatorial optimization problems such as traveling salesman problem [10]. However, there have been few studies on complex-valued neural networks for solving optimization problems so far. Thus, we consider graph coloring problems as a possible application of a complex-valued network in this study. In vertex coloring of a given graph, the purpose is to find a way of coloring the vertices of the graph such that no two adjacent vertices share the same color. Graph coloring is a classic constraint satisfaction problem with realistic applications such as time-tabling, short circuit testing in printed circuits, VLSI design, and register allocation in a compiler.

Section 2 describes the definition of graph coloring and reviews a conventional energy function-based approach to the problem. A heuristic algorithm based on a complex-valued neural network is introduced in Sec. 3, and then applied to benchmarks of graph coloring in Sec. 4. Summary is given in Sec. 5.

2. Graph Coloring with Neural Networks

2.1. Vertex Coloring

Graph coloring is a special case of graph labeling. We limit our focus to a vertex coloring of a planar graph in this study. Vertex coloring is to assign colors to vertices of a given graph so that no pair of adjacent vertices having a common edge share the same color. A coloring using at most K colors is called a (proper) K-coloring and is equivalent to the problem of partitioning the vertex set into K or fewer independent sets. Minimum coloring problem asking for the smallest number of colors needed to color the graph is NP-hard, while the corresponding decision problem (vertex coloring) in the general case is NP-complete.

Let us consider an undirected graph $G = \{V, E\}$ composed of a set of $N = |V(G)|$ vertices and a set of $M = |E(G)|$ edges. The set of vertices is denoted by $V = \{v_i,\ i = 1, \ldots, N\}$ and the set of edges is by $E = \{e_{ij} \mid v_i \text{ and } v_j \text{ are connected}\}$. A K-coloring of G is a partition of V into K sets (color classes) $V_1, \ldots, V_k$ such that $\forall e_{ij}, v_i \in V_l \rightarrow v_j \notin V_l$.

2.2. Energy Function-Based Approaches

There have been energy-function based approaches to graph coloring [11, 12]. The previous approach [12] is briefly introduced here for comparison with our method. For a graph G with N nodes, a network composed of N neurons is constructed. An adjacency matrix $A = (a_{ij})$ has the components as follows:

$$a_{ij} = \begin{cases} 1 & \text{if} \quad e_{ij} \in E \\ 0 & \text{if} \quad e_{ij} \notin E \end{cases}. \tag{1}$$

The color assigned to i-th node is represented by the state of i-th neuron, i.e., $s_i \in \{1, \ldots, K\}$. Each neuron receives the summation of the inputs from the states of all the adjacent neurons. A specific energy function is defined for a specific problem such that detection of an optimal solution is reduced to minimization of the energy function.

The energy function for vertex coloring is given as follows:

$$E_r(\mathbf{s}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \delta(s_i, s_j), \tag{2}$$

where a color assignment to all vertices of G is denoted by $\mathbf{s} = (s_1, \ldots, s_N)^t$. The function $\delta(\cdot, \cdot)$ is the Kronecker delta function, i.e., $\delta(i, j) = 1$ if $i = j$, 0 otherwise. The energy function (2) is equivalent to the number of violations of the constraint, that is, the number of edges with improperly colored vertices. Therefore, the goal is reduced to minimization of the energy function $E_r(\mathbf{s})$ by updating the neuronal states $\mathbf{s}$.

For a randomly selected index m and a new candidate state $c \in \{1, \ldots, K\}$, we consider a new network state $\mathbf{s}'$ where $s'_m = c$ and $s'_i = s_i$ for $i \neq m$ and evaluate the difference of the energy function, i.e., $\Delta E_r = E_r(\mathbf{s}') - E_r(\mathbf{s})$. If and only if $\Delta E_r < 0$, the network state is updated from $\mathbf{s}$ to $\mathbf{s}'$. By iterating the above procedures, the energy function necessarily reaches a local minimum of the energy landscape. This greedy algorithm is quite effective for finding a better coloring in combination with multiple random restarts.

3. Complex-Valued Neural Network

The complex-valued Hopfield network proposed by Jankowski *et al.* [6] is composed of N complex-valued neurons whose states are defined on the unit circle in the complex domain. The network state of K-valued neurons is represented by a complex vector $\mathbf{z} = (z_1, \ldots, z_N)^t$ where $z_n = \exp(ik_n\theta_K)$ with $\theta_K = 2\pi/K$ and $k_n \in \{0, 1, \ldots, K-1\}$ for $n = 1, 2, \ldots, N$. When $K = 2$, the complex-valued network is reduced to the binary Hopfield network [5].

The energy function of the complex-valued Hopfield network is given as follows:

$$E_c(\mathbf{z}) = -\frac{1}{2} \sum_{n=1}^{N} \sum_{j=1}^{N} w_{nj} \bar{z}_n z_j, \tag{3}$$

which is real-valued if $W = (w_{nj})$ is an Hermitian matrix. It is guaranteed that the energy function value is monotonically decreasing for each update of a neuronal state as follows:

$$z'_n = \text{csign}_K \left(e^{i\theta_K/2} \sum_{j=1}^{N} w_{nj} z_j \right) \tag{4}$$

where z'_n is the updated state of the n-th neuron and the weighted sum of inputs multiplied by the rotational factor is given to the activation function. The complex-signum function csign_K is defined as follows:

$$\text{csign}_K(z) = \begin{cases} e^0 & 0 \leq \arg(z) < \theta_K, \\ e^{i\theta_K} & \theta_K \leq \arg(z) < 2\theta_K, \\ \vdots & \vdots \\ e^{i\theta_K(K-1)} & (K-1)\theta_K \leq \arg(z) < 2\pi. \end{cases} \tag{5}$$

In our previous work [8], the update scheme (4) is rewritten as follows:

$$z'_n = r_K \circ f_K \circ q_K \left(\sum_{j=1}^{N} w_{nj} z_j \right), \tag{6}$$

where $r_K(x) = \exp(ix\theta_K)$ for $x \in [0, K)$, $f_K(x) = [x]$ for $x \in [0, K)$, and $q_K(z) = \arg(z)/\theta_K$. Then, in order to improve this model, a multilevel step function f_K is replaced by its nonlinear version:

$$m_K(x) = \left(\sum_{i=0}^{K+1} g(x - i) \right) - \frac{1}{2} \pmod{K}, \tag{7}$$

where $g(x) = 1/(1 + \exp(-x/\epsilon))$. This multilevel sigmoid function, which is used as an activation function in generalized Hopfield networks [13, 14], is incorporated into the complex-valued activation function. The modification is quite reasonable because the performance of conventional neural networks has often been enhanced by a factor of nonlinear dynamics. The update rule is given as follows:

$$z'_n = r_K \circ m_K \circ q_K \left(\sum_{j=1}^{N} w_{nj} z_j \right), \tag{8}$$

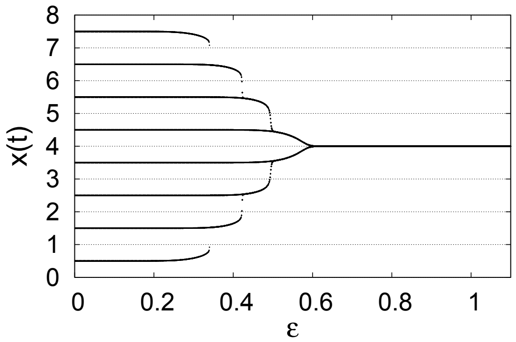

Figure 1: Bifurcation diagram of the multilevel sigmoid function map (9) with $K = 8$.

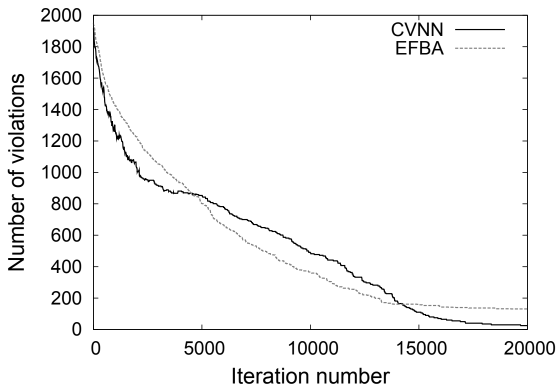

Figure 2: The transition of the violations with the minimization of the energy function of the energy function-based approach (EFBA) and the complex-valued neural network (CVNN) for le450_5c.

where $r_K \circ m_K \circ q_K$ denotes the complex-sigmoid function. Here the rotational factor for the weighted sum of inputs is not necessary because the multilevel function is already shifted. The neurons are updated asynchronously. The complex-sigmoid function approximates the csign_K function in the limit of $\epsilon \to 0$. In this sense, the network with the update scheme (8) is a generalization of that with the complex-signum activation function [6]. The output state of z_n is expressed as $\exp(ik_n\theta_K)$ by discretization with the csign_K function.

In order to understand the effect of the nonlinearity of the function (7), let us consider the discrete-time dynamical system as follows:

$$x(t+1) \quad = \quad m_K(x(t)). \quad t = 0, 1, \ldots \tag{9}$$

Figure 1 shows the bifurcation diagram of the map (9) with variation of the value of the nonlinearity parameter ϵ. For a small value of ϵ, K stable fixed points are coexisting. The multiple stable states merge into a smaller number of stable states with increase of ϵ. Therefore, we need to set ϵ at an appropriate value for representing a K-valued state with a single complex-valued neuron.

For solving a graph coloring problem, the adjacency matrix is associated with the weight matrix, i.e., $W = -A$. Since W is real-valued and Hermitian,

$$E_c(\mathbf{z}) = \frac{1}{2}\sum_{n=1}^{N}\sum_{j=1}^{N} a_{nj}\cos((k_j - k_n)\theta_K). \tag{10}$$

where the cosine term takes a maximum value 1 if and only if $k_j = k_n$. Therefore, minimizing the energy function E_c has an effect on decreasing the number of edges which share the same color vertices. However, it should be noted that an optimal coloring is not necessarily corresponding to $E_c = 0$.

4. Numerical Experiments

Vertex coloring problems are solved by a heuristic method based on the complex-valued neural network (CVNN). Our purpose is to investigate the effect of the nonlinearity parameter on the optimization performance and compare it with the energy function-based approach (EFBA). The two methods were implemented by C, compiled with the GNU gcc compiler. The algorithm was tested on the official benchmarks from the 1993 DIMACS graph coloring challenge [15]. The initial states of the neurons were determined randomly.

Figure 2 shows the transition of the number of violations with repeated updates of neuronal states in optimization of the problem le450_5c. We can see that the proposed method is better than the conventional approach in terms of the quality of the finally obtained solution. The computation time until convergence is almost the same between the two methods. It is obvious in Fig. 2 that the EFBA is based on the monotonically decreasing energy function while the CVNN permits an increase of the energy level.

With variation of the nonlinearity parameter ϵ, the performance of the optimization with the CVNN is variable. After trial-and-error testing with different values of ϵ, we fix ϵ at 0.2 in the numerical experiments. Table 1 shows the comparison results between the CVNN and the EFBA in some benchmarks. In all the simulations, the maximum number of updates is fixed. For problems with smaller K (le450_5a and le450_5c), the CVNN is better than or comparable to the EFBA. The EFBA is better than the CVNN for other problems. The difference between the two methods becomes larger for the problems with a larger value of K. The reason of this fact may be due to the multiplicity of the cosine term in (10) in the complex-valued network. Namely, $\cos((k_j - k_n)\theta_K)$ takes a component of the set $\{1, \cos(\theta_K), \ldots, \cos((K-1)\theta_K)\}$. Therefore, if the two connected vertices do not have the same color, the corresponding cosine term is biased depending on the pair of the two colors. The bias becomes diverse as K increases. A future issue to be considered includes elimination of the bias for better performance of the CVNN.

Table 1: Simulation results on the 1993 DIMACS Graph Coloring Challenge [15] with the complex-valued neural network (CVNN) and the energy function-based approach (EFBA) [12]. The number of violations is indicated by v.

Graph				CVNN		EFBA	
name	$N = \|V\|$	$M = \|E\|$	K	v	$v/M\%$	v	$v/M\%$
le450_5a	450	5714	5	583.4	10.2	584.0	10.2
le450_5c	450	9803	5	22.0	0.22	80.8	0.82
le450_15a	450	8168	15	268.0	3.28	128.2	1.57
le450_15c	450	16680	15	742.4	4.45	567.8	3.40
le450_25a	450	8260	25	314.0	3.80	23.4	0.28
le450_25c	450	17343	25	834.4	4.81	188.0	1.08

5. Summary

In the present study, graph coloring problem has been solved by a complex-valued neural network for multivalued information processing. Instead of minimizing the objective function corresponding to the number of edges connecting nodes with same colors, we have considered a relaxation problem which is to minimize a complex-valued energy function. Our simulation has shown that in some benchmarks the heuristic algorithm based on the complex-valued network has brought about better solutions compared with a conventional energy function-based approach.

Acknowledgments

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