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A Novel Resonate-and-Fire-type Digital Spiking Neuron and its Bifurcation Analysis

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Abstract—In this paper we propose a novel resonate-and-fire-type digital spiking neuron. The neuron accepts an stimulation input and generates various firing spike-trains. We derive a return map which can describe the neuron dynamics without any approximations. The map is a continuous/discrete hybrid map. Using the map we analyze the neuron dynamics and clarify typical bifurcation mechanisms of the neuron. We also discuss relations between responses of our model and an ODE spiking neuron model.

1. Introduction

Until now many spiking neuron models have been proposed [1]-[9] in order to develop artificial pulsed neural networks and to explore biological neuron dynamics. As shown in Table 1, the integrate-and-fire model has 1-dimensional continuous state dynamics with reset [1],[2],[4]. The resonate-and-fire model has 2-dimensional continuous state dynamics with reset [1],[2],[5]. They are analog models, so dynamic parameter adjustments are difficult after implementation.

On the other hand, the digital spiking neuron model which has digital state dynamics has been proposed [7]-[9]. The model *can change its parameter values on chip after implementation*. In this paper, we propose a novel resonate-and-fire-type digital spiking neuron and analyze it using a return map without approximations. We also discuss relations between responses of our model and an ODE spiking neuron model. Significances and novelties of this paper include the following points. (a) The new model can exhibit various neural-behaviors such as the integrate-and-fire and the resonate-and-fire behaviors whereas our old model [7]-[9] can exhibit the integrate-and-fire behavior only. (b) The new model can be implementation as coupled shift registers and then parameter values (e.g., wiring pattern among the registers) can be dynamically adjusted after implementation.

2. Digital Spiking Neuron

In this paper, we present a novel resonate-and-fire-type digital spiking neuron. As shown in Figure 1, the neuron consists of three parts: N pieces of u-cells that are indexed by $i \in \{0, 1, \dots, N-1\}$ (left part); M pieces of v-cells that are indexed by $j \in \{0, 1, \dots, M-1\}$ (right part); and

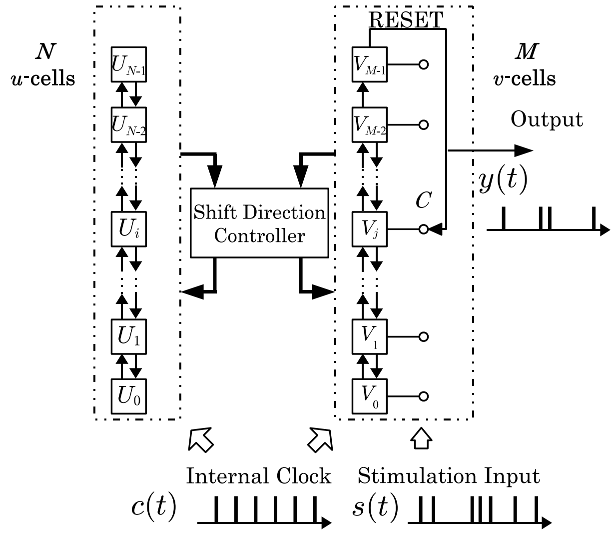

Figure1: Digital spiking neuron.

a shift direction controller (middle part). The u-cells and the v-cells correspond to a recovery variable and a membrane potential of the Izhikevich's simple model [1], respectively. Each u-cell has a binary state $U_i(t) \in \{0, 1\}$, where $t \in [0, \infty)$ is a continuous time. We assume that one u-cell has a state "1" and the other u-cells have states "0". Then we can introduce the integer state $u(t) = i$ if $U_i(t) = 1$, where $u(t) \in \{0, 1, \dots, N-1\}$. Similarly, each v-cell has a binary state $V_j(t) \in \{0, 1\}$, one v-cell has a state "1", and the other v-cells have states "0". Then we can introduce the integer state $v(t) = j$ if $V_j(t) = 1$, where $v(t) \in \{0, 1, \dots, M-1\}$. As shown in Figure 1, the stimulation input $s(t) \in \{0, 1\}$ and the internal clock $c(t) \in \{0, 1\}$ change the states (v, u) of the neuron. The stimulation input $s(t)$ is applied to the v-cells only and shifts the state v

Table1: Difference between the proposed neuron and other models. IF: integrate-and-fire, RF: resonate-and-fire, L:linear, NL: non-linear, NA: non-autonomous.

Models	Dynamics	Behavior	Dynamic parameter adjustment
IF [1],[2],[4]	1-D ODE + reset	IF, NA IF	×
RF [1],[2],[5]	2-D L-ODE + reset	IF, RF	×
Izhikevich[1],[2]	2-D NL-ODE + reset	IF, RF	×
DSN [7]-[9]	Discrete state + reset	NA IF	○
RF-type DSN (proposed here)	Discrete state + reset	IF, RF	○

Table2: Control rules. ↑:shift up, ↓:shift down, $M_c := \left\lfloor \frac{M-1}{2} \right\rfloor$. $f(v)$ is a function which determines the shift direction.

Subspaces	u-cells	v-cells
$\mathbf{A}_0 := \{(v,u) \mid v = M_c, u = f(v)\}$	fixed	fixed
$\mathbf{A}_1 := \{(v,u) \mid v > M_c, u \geq f(v)\}$	↑	↓
$\mathbf{A}_2 := \{(v,u) \mid v \leq M_c, u > f(v)\}$	↓	↓
$\mathbf{A}_3 := \{(v,u) \mid v < M_c, u \leq f(v)\}$	↓	↑
$\mathbf{A}_4 := \{(v,u) \mid v \geq M_c, u < f(v)\}$	↑	↑

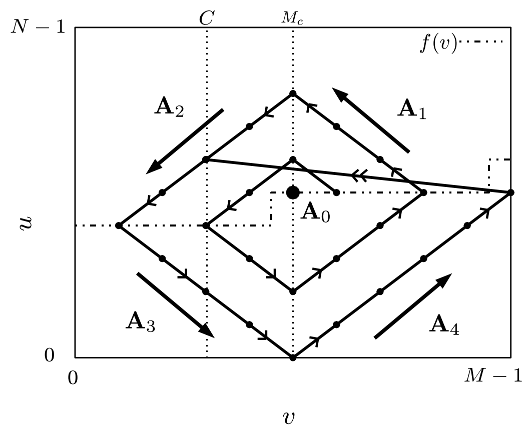

Figure2: Shift directions and typical orbit.

upward. Also the v-cells have a reset mechanism, i.e., the state v is reset to a constant value C if the state v reaches $M-1$ (firing threshold) and an internal clock $c(t) = 1$ arrives. At the reset moment, a firing spike $y(t) = 1$ is generated. The internal clock $c(t)$ shifts the states (v, u). The shift direction is decided by the shift direction controller as shown in Figure 1. The controller rules the shift direction based on the subspaces $\{\mathbf{A}_0, \ldots, \mathbf{A}_4\}$ of the entire space $\{(v, u) \mid 0 \leq v < M, 0 \leq u < N\}$ as shown in Table 2. The control rule realizes a rotation dynamics (resonation-like dynamics) as shown in Figure 2. Repeating the rotation and firing dynamics, the neuron generates a spike-train $y(t)$.

3. Return Map and Analysis

3.1. Return map

In this section we derive a return map to analyze the neuron. As shown in Figure 3, we assume that the internal clock $c(t)$ and the stimulation input $s(t)$ are periodic, and these periods are 1 and $d \in [0, \infty)$, respectively. We define $t = 0$ as the time when the internal clock $c(t)$ is applied first, and a variable $\theta \in (0, d] =: \mathbf{\Theta}$ as the initial phase of the stimulation input $s(t)$. Let $t_1 > 0$ and $t_2 > t_1$. We define a function $I_{pt}(t_1, t_2)$ as the number of the spikes of the stimulation input $s(t)$ during $t_1 < t \leq t_2$ as follows.

$$I_{pt}(t_1, t_2) = \begin{cases} \lfloor (t_2 - \theta)/d \rfloor - \lfloor (t_1 - \theta)/d \rfloor & \text{if } t_1 \geq \theta. \\ \lfloor (t_2 - \theta)/d \rfloor + 1 & \text{if } t_2 \geq \theta > t_1. \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

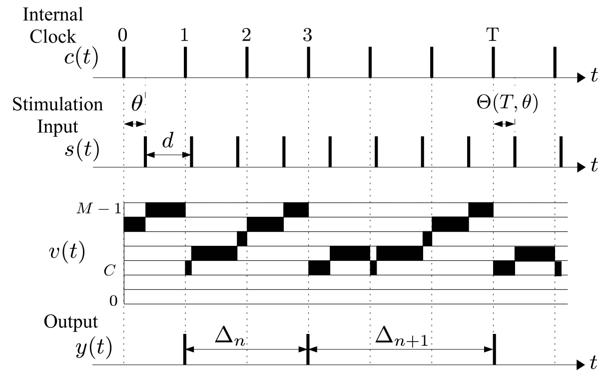

Figure3: Sketch of the neuron dynamics and meaning of variables.

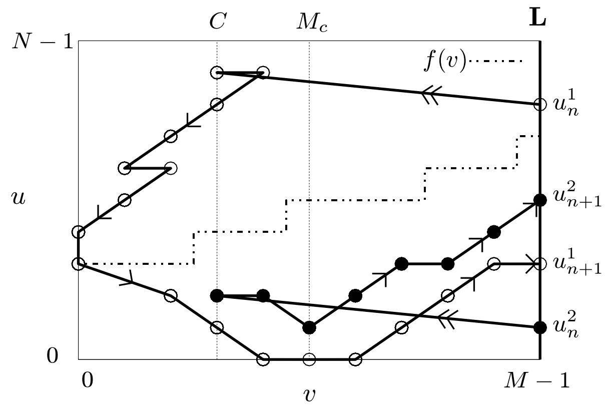

Figure4: Definition of return map, each u_n^k map to u_{n+1}^k.

Additionally, we define a variable $v'(t)$ and functions $x(v, u)$ and $z(v, u)$ as follows.

$$v'(t) := v(t) + I_{pt}(t, t+1).$$

$$x(v,u) := \begin{cases} 1 & \text{if } (v,u) \in \mathbf{A}_3 \cup \mathbf{A}_4. \\ -1 & \text{if } (v,u) \in \mathbf{A}_1 \cup \mathbf{A}_2, v \neq 0. \\ 0 & \text{otherwise.} \end{cases}$$

$$z(v,u) := \begin{cases} 1 & \text{if } (v,u) \in \mathbf{A}_1 \cup \mathbf{A}_4, u \neq N-1. \\ -1 & \text{if } (v,u) \in \mathbf{A}_2 \cup \mathbf{A}_3, u \neq 0. \\ 0 & \text{otherwise.} \end{cases}$$

Then we can derive a discrete time map which represents the dynamics of the neuron for a discrete time $\tau \in \{0, 1, 2, \ldots\} =: \mathbf{T}$ as follows.

$$\begin{cases} v(\tau+1) = v'(\tau) + x(v'(\tau), u(\tau)) \\ u(\tau+1) = u(\tau) + z(v'(\tau), u(\tau)) \end{cases} \quad \text{for } v'(\tau) < M-1. \quad (2)$$

$$\begin{cases} v(\tau+1) = C \\ u(\tau+1) = u(\tau) + z(v'(\tau), u(\tau)) \end{cases} \quad \text{for } v'(\tau) \geq M-1. \quad (3)$$

Next, we derive a return map on firing threshold $\mathbf{L} = \{(v, u) \mid v = M-1, 0 \leq u < N\}$ as shown in Figure 4. Let a point on $\mathbf{L}$ be represented by its u-coordinate. The return map consists of two variables: $u_n \in \mathbf{L}$ means the starting point of $u(\tau)$ from $\mathbf{L}$; $\theta_n \in \mathbf{\Theta}$ means a phase of the stimulation input $s(t)$ with respect to the starting moment, where $n \in \mathbf{T}$. Hence the return map is two-dimensional. Let the states (v, u) be in the threshold $\mathbf{L}$. Then the time τ is reset to zero. From Equation (3) we have

$$\begin{cases} v(0) = C, \\ u(0) = u_n + z(M-1, u_n), \end{cases}$$

and using Equation (2) and (3), the states (v, u) are determined uniquely for all τ. And we denote an inter spike interval (ISI) by Δ_n (see Figure 3). The ISI Δ_n is given by

$$\Delta_n = H(u_n, \theta_n) = \min\{\tau + 1 \mid \tau \in \mathbf{T}, v'(\tau) \geq M - 1\}, \quad (4)$$
$$H : \mathbf{L} \times \mathbf{\Theta} \to \{1, 2, \ldots\},$$

where $v'(\tau)$ is decided by (u_n, θ_n), so Δ_n is a function of (u_n, θ_n). And we define a function $\Theta(\tau, \theta)$ as follows.

$$\Theta(\tau, \theta) = \min\{\theta + ld - \tau \mid l, \tau \in \mathbf{T}, \theta + ld > \tau\}. \quad (5)$$

$\Theta(\tau, \theta)$ denotes a phase of the stimulation input $s(t)$ with respect to the moment τ as shown in Figure 3. Finally we can derive a return map as follows.

$$\begin{aligned} u_{n+1} &= F(u_n, \theta_n) = u(\Delta_n - 1), \\ \theta_{n+1} &= G(u_n, \theta_n) = \Theta(\Delta_n, \theta_n), \end{aligned} \quad (6)$$
$$F : \mathbf{L} \times \mathbf{\Theta} \to \mathbf{L}, \quad G : \mathbf{L} \times \mathbf{\Theta} \to \mathbf{\Theta}.$$

As a result, the dynamics of the neuron is described by the return map Equation (6) and the function $H(u_n, \theta_n)$ which gives the ISI Δ_n without approximations.

3.2. Analysis of Typical Bifurcation Mechanism

Using the return map in Equations (6), we analyze the dynamics of the neuron. Examples of the projection $F(u_n, \theta_n)$ of the return map (6) are shown in Figure 5. We introduce the following.

Definition 1 *Let* $r \geq 2$ *be a positive integer. Sets* $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{r-1})$ *are said to be period-r sets if they are the minimum disjoint sets such that* $u_n \in \mathbf{u}_{n+q \pmod r}$ *for all* $n \geq 1$ *and for a fixed integer* $q \in \{0, 1, \ldots, r-1\}$.

In Figure 5(a) there are *period-2 sets* $(\mathbf{u}_0, \mathbf{u}_1)$. The sets $(\mathbf{u}_0, \mathbf{u}_1)$ are very close to each other and seems like a *period-1 set*, so the output $y(t)$ seems to be 1-periodic. It corresponds to "tonic spiking" of a neuron model [1].

In Figure 5(b) there are still *period-2 sets* $(\mathbf{u}_0, \mathbf{u}_1)$. But the distance between the sets $\mathbf{u}_0$ and $\mathbf{u}_1$ is longer than that in Figure 5(a), so the output $y(t)$ is 2-periodic.

In Figure 5(c), as the map changes, the *period-2 sets* turns into *period-3 sets*, so the output $y(t)$ is 3-periodic. In this case the return map has other periodic sets (not shown in the figure), i.e., the return map has co-existing attractors with respect to the initial states.

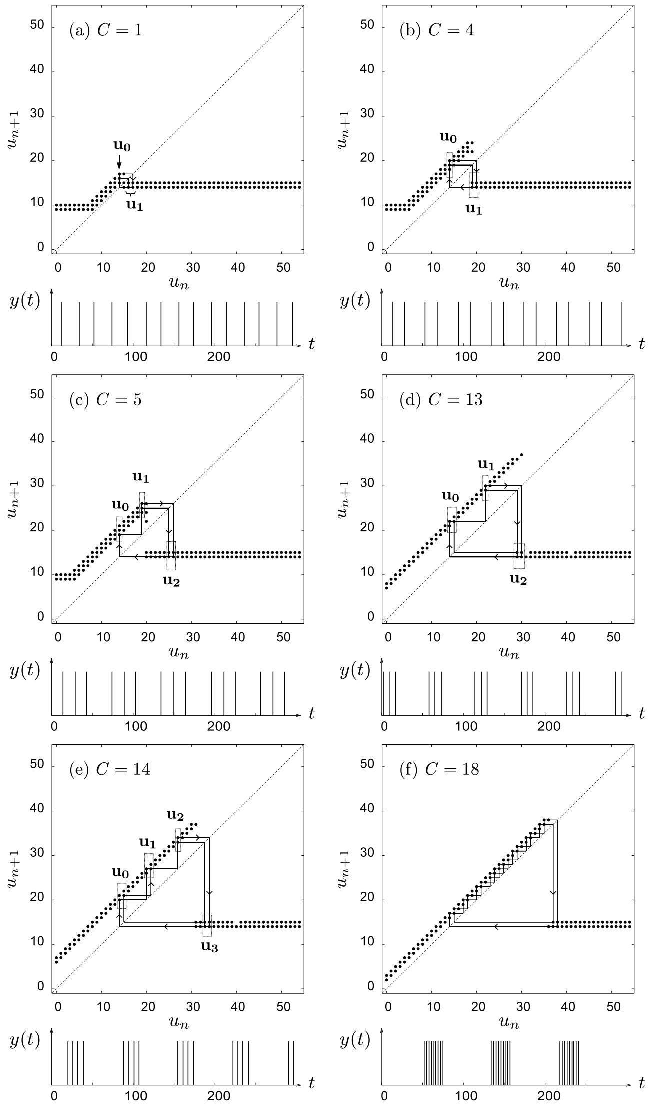

Figure5: Projection $F(u_n, \theta_n)$ of the return map with typical orbits and the output spike-train $y(t)$. Parameters are $(M, N) = (21, 55)$, $f(v) = \left\lfloor 1.3(v - M_c) + \left\lfloor \frac{N-1}{2} \right\rfloor \right\rfloor$ and $d = 10.1$.

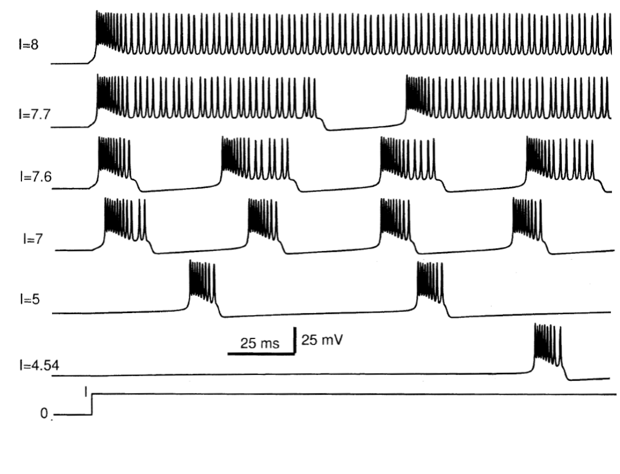

Figure6: "Tonic spiking ↔ bursting" bifurcation in $I_{\mathrm{Na,p}} + I_{\mathrm{K}} + I_{\mathrm{K(M)}}$ model [1]. I denotes a injected DC current.

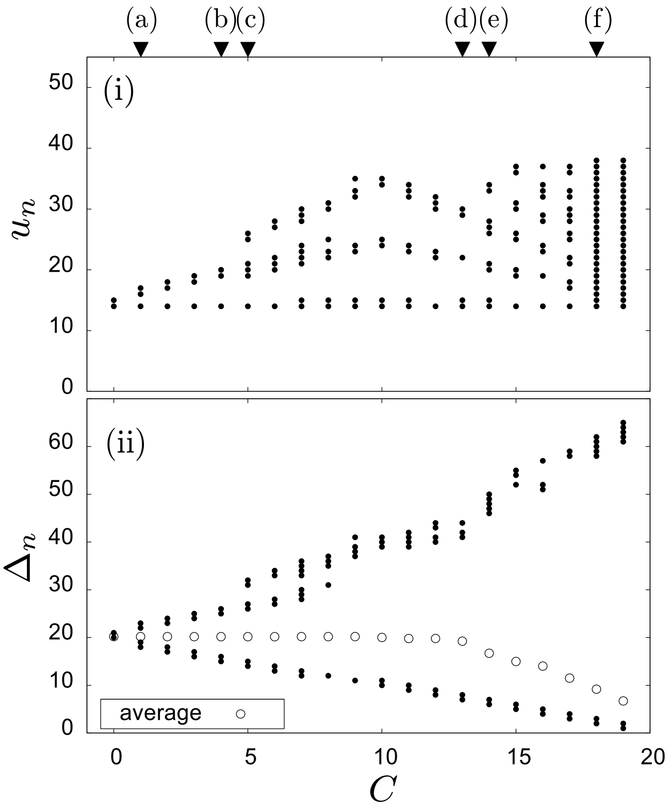

Figure7: (i) Bifurcation diagram of the state u_n for parameter C. (ii)Characteristics of ISI Δ_n. All parameters are the same as that in Figure 5.

In Figure 5(d), there are still *period-3 sets* like Figure 5(c). However, the spike-train $y(t)$ in Figure 5(d) is more like a bursting spike-train than that in Figure 5(c).

Increasing the parameter C further, the left part of the map $F(u_n, \theta_n)$ (located on a line) approaches to a diagonal line $u_{n+1} = u_n$, and then the number of burst spikes increases as shown in Figure 5(d)-(f). In Figure 5(e) the *period-3 sets* turns into *period-4 sets* so the output $y(t)$ is 4-periodic. In Figure 5(f), the spike-train $y(t)$ has many burst spikes. This phenomena corresponds to the "tonic bursting" of the neuron model [1].

The response of our model from the tonic spiking in Figure 5(a) to the tonic bursting in Figure 5(f) may correspond to the "tonic spiking ↔ bursting" bifurcation shown in Figure 6 [1]. Figure 7 shows a bifurcation diagram of the state u_n and characteristics of ISI Δ_n. The points (a)-(f) in Figure 7 correspond to the maps $F(u_n, \theta_n)$ in Figure 5(a)-(f), respectively. Detailed analysis on relation between the responses and the bifurcation is an important future problems.

4. Conclusions

In this paper, we have proposed the novel resonate-and-fire-type digital spiking neuron and clarified its typical bifurcation mechanisms using the return map. The neuron can exhibit various neural-behaviors (e.g., tonic spiking and tonic bursting as in Section 3). Here we can summarize some comparisons of the proposed neuron with other models as in Table 3. As shown in this table, the proposed model can exhibit as much neural-behaviors as other models. Future problems include: (a) analysis for another parameters, (b) FPGA implementation of the neuron, (c) proposing a learning algorithm, and (d) development of network of the proposed neurons.

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Table3: Comparison of properties of the spiking neuron models. TS: tonic spiking, TB: tonic bursting, IT: integrator, RS: resonator, BI: biophysically meaningful.

Models	TS	TB	IT	RS	BI
Integrate-and-fire	○	×	○	×	×
Resonate-and-fire	○	×	○	○	×
Izhikevich	○	○	○	○	×
FitzHugh-Nagumo	○	×	×	○	×
Moris-Lecar	○	×	○	○	○
The neuron in this paper	○	○	○	○	×