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Analysis of Dynamical Characteristic of Particle Swarm Optimization

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Abstract—A particle swarm optimization (PSO) is one of the powerful systems for solving global optimization problems. The searching ability of such PSO depends on the inertia weight coefficient, and the acceleration coefficients. Since the acceleration coefficients are multiplied by a random vector, the system can be regarded as a stochastic system. In order to analyze the dynamics rigorously, we pay attention to deterministic PSO which does not contain any stochastic factors. Especially, we propose a canonical deterministic PSO (CD-PSO). First, we compare with the searching ability between the standard PSO and the CD-PSO. The deterministic PSO system must converge to the fix point without divergence. On the other hand, the standard PSO may diverge depending on the random parameter. Due to this divergence property, the standard PSO has high performance compared to the deterministic PSO. To overcome this weakness of the deterministic PSO, we proposed the reacceleration procedure for the deterministic PSO. In this article, the reacceleration operation is applied to the CD-PSO. Also, we confirm the performance of the CD-PSO with the reacceleration operation.

1. Introduction

Searching for an optimal value of a given evaluation function is very important. In order to solve such optimization problems speedily, various heuristic optimization algorithms have been proposed. Particle swarm optimization (PSO), which was originally proposed by J. Kennedy and R. Eberhart [1],[2], is one such heuristic algorithm. The PSO algorithm is a useful tool for optimization problems[3]-[6].

The original PSO is described as

$$\boldsymbol{v}_j^{t+1} = w\boldsymbol{v}_j^t + c_1\boldsymbol{r}_1(\boldsymbol{pbest}_j^t - \boldsymbol{x}_j^t) + c_2\boldsymbol{r}_2(\boldsymbol{gbest}^t - \boldsymbol{x}_j^t) \quad (1)$$

$$\boldsymbol{x}_j^{t+1} = \boldsymbol{x}_j^t + \boldsymbol{v}_j^{t+1} \quad (2)$$

where $w \geq 0$ is an inertia weight coefficient, $c_1 \geq 0$, and $c_2 \geq 0$ are acceleration coefficients, and $\boldsymbol{r}_1 \in [0,1]^N$, and $\boldsymbol{r}_2 \in [0,1]^N$ are N-dimensional uniformly distributed random vectors whose each component is in the range $[0,1]$. $\boldsymbol{x}_j^t \in \Re^N$ denotes the location of the j-th particle on the t-th iteration in the N-dimensional space, and $\boldsymbol{v}_j^t \in \Re^N$ denotes a velocity vector of the j-th particle on the t-th iteration. $\boldsymbol{pbest}_j^t \in \Re^N$ means the location that gives the best value of the evaluation function of the j-th particle on the t-th iteration. $\boldsymbol{gbest}^t \in \Re^N$ means the location which gives the best value of the evaluation function on the t-th iteration in the swarm.

The particles in the swarm fly through the N-dimensional space according with Eqs. (1) and (2). Each particle shares information of a current optimal value of the evaluation function and its corresponding location of the best particle. Also, each particle memorizes its record of the best evaluation value and its best location. On the basis of such information, the moving direction and velocity are calculated by Eq. (1). Namely, all particles will move toward a coordinate that gives the current best value of the evaluation function.

The searching ability of such PSO depends on the inertia weight coefficient, and the acceleration coefficients. Since the acceleration coefficients are multiplied by a random vector, the system can be regarded as a stochastic system. In order to analyze the dynamics of such PSO, M. Clerc, and J. Kennedy proposed a simple deterministic PSO system (D-PSO), and analyzed its dynamics theoretically[4]. The D-PSO does not contain stochastic factors, namely, the random coefficients have been omitted from the original PSO system. The analysis of such D-PSO is very important for determining the effective parameters of the standard PSO[4]-[5]. The dynamics of the D-PSO depends on the eigenvalues of the system[7]. We focus on this fact, we proposed a canonical deterministic PSO (CD-PSO) which is tranformed from the simple deterministic PSO[7]. According to the results, the dynamics depends on the eigenvalues of the system[7]. In this article, we compare with the searching ability between the simple PSO and the CD-PSO.

For simplicity, we choose the stable parameters for the stability of the D-PSO. Therefore, the D-PSO system must converge to the fix point without divergence. On the other hand, the standard PSO may diverge depending on the random parameter. Due to this divergence property, the standard PSO has high performance compared to the D-PSO. To overcome this weakness of the D-PSO, we proposed the reacceleration procedure for the D-PSO. In this article, the reacceleration operation is applied to the CD-PSO. Also, we confirm the performance of the CD-PSO with the reacceleration operation.

2. Canonical deterministic PSO

In this section, we introduce a canonical deterministic PSO system. Since each dimension variable of the particle is independent, we can consider one dimensional case without loss of generality. Therefore, we consider one dimensional system hereafter. The CD-PSO is based on one dimensional D-PSO which is described as the following matrix form.

$$\begin{bmatrix} v_j^{t+1} \\ y_j^{t+1} \end{bmatrix} = \begin{bmatrix} w & -c \\ w & 1-c \end{bmatrix} \begin{bmatrix} v_j^t \\ y_j^t \end{bmatrix} \tag{3}$$

where $y_j^t = x_j^t - p_j^t$ is a normalized position variable and $c = c_1 + c_2$ is a generalized acceleration coefficient. $\boldsymbol{p}_j^t$ denotes a desired fixed point which is described as

$$\boldsymbol{p}_j^t = \frac{c_1}{c}\boldsymbol{pbest}_j^t + \frac{c_2}{c}\boldsymbol{gbest}^t \tag{4}$$

The dynamics of the D-PSO is governed by the eigenvalues of the matrix in Eq. (3). In this article, we consider the case where the eigenvalue λ is complex conjugate number. The eigenvalue λ is given as

$$\lambda = \frac{1-c+w}{2} \pm j\frac{\sqrt{4w-(1-c+w)^2}}{2}. \tag{5}$$

Based on this eigenvalue, we can derive the following coordinate conversion.

$$\begin{bmatrix} v_j^t \\ y_j^t \end{bmatrix} = \boldsymbol{P}\begin{bmatrix} \tilde{v}_j^t \\ \tilde{y}_j^t \end{bmatrix} \tag{6}$$

where

$$\boldsymbol{P} = \sqrt{\frac{2}{\sqrt{4w-(1-c+w)^2}}}\begin{bmatrix} 1 & 0 \\ \frac{w+c-1}{2} & \frac{\sqrt{4w-(1-c+w)^2}}{2c} \end{bmatrix}$$

By using the matrix $\boldsymbol{P}$, we derive the following relation.

$$\begin{bmatrix} \delta & \omega \\ -\omega & \delta \end{bmatrix} = \boldsymbol{P}^{-1}\begin{bmatrix} w & -c \\ w & 1-c \end{bmatrix}\boldsymbol{P}$$

Therefore, we derive a canonical form of Eq. (3) as the following.

$$\begin{bmatrix} \tilde{v}_j^{t+1} \\ \tilde{y}_j^{t+1} \end{bmatrix} = \begin{bmatrix} \delta & \omega \\ -\omega & \delta \end{bmatrix}\begin{bmatrix} \tilde{v}_j^t \\ \tilde{y}_j^t \end{bmatrix} \tag{7}$$

The transformed coordinate $(\tilde{v}_j^t, \tilde{y}_j^t)$ is regarded as the original coordinate of the given evaluation function.

3. Reacceleration

The finding solution ability is related to the velocity variation of the particle. The velocity of the standard PSO may diverge depending on the acceleration parameters, therefore, the standard PSO sets the limitation of the velocity.

Table 1: Benchmark functions

Function	Optimum value
Sphere	
$f_1(x) = \sum_{d=1}^{N} x_d^2$	$f_1(0,0,0,\ldots,0) = 0$
Rosenbrock	
$f_2(x) = \sum_{d=1}^{N-1}(100(x_{d+1} - x_d^2)^2 + (x_d - 1)^2)$	$f_2(1,1,1,\ldots,1) = 0$
Rastrigin	
$f_3(x) = 10N + \sum_{d=1}^{N}((x_d)^2 - 10\cos(2\pi x_d))$	$f_3(0,0,0,\ldots,0) = 0$
Griewank	
$f_4(x) = 1 + \frac{1}{4000}\sum_{d=1}^{N} x_d^2 - \prod_{d=1}^{N}\cos\left(\frac{x_d}{\sqrt{d}}\right)$	$f_4(0,0,0,\ldots,0) = 0$
Schaffer's $f6$	
$f_5(x) = 0.5 + \frac{\left(\sin\left(\sqrt{x_d^2 + x_{d+1}^2}\right)\right)^2 - 0.5}{\left(1.0 + 0.001\left(x_d^2 + x_{d+1}^2\right)\right)^2}$	$f_5(0,0) = 0$

By using the effect of this divergence propserty, the searching ability of the standard PSO is improved. On the other hand, the parameters of the D-PSO satisfy the stable condition, therefore, the particle of the D-PSO converges to the fixed point without the divergence. This is one reason why the performance of the CD-PSO is worsened.

To overcome this weakness of the D-PSO, we propose a reacceleration procedure for the velocity. Such approaches have been proposed. For instance, Inertia Weights Approach (IWA) is well known approach of the parameter setting that shifts from global search to the concentrative search. IWA achieves better searching results for the low-dimensional problems by the decreasing Inertia Weight with iterations. Velocity-based Reinitialization (VBR) improves the searching solution[8]. This technique is paid attention to the velocity variation. When the velocity of swarm is dipped from certain stagnation threshold, VBR procedure initializes the position of all particle of swarm. IWA and VBR are operated to ”swarm”. These procedures are very simple, however, the detailed operation is seems to difficult.

On the other hand, our proposed procedure is very simple. The velocity accelerates again when the velocity dips from the given lower bound value. We note that the velocity of each dimension of the particle is calculated independently. Thus, we set the lower limit for each dimension. If the velocity is dipped from the lower limit, the velocity is reset, namely, the velocity is reaccelerated. We call the above proposed procedure ”Reacceleration”. By using ”Reacceleration”, Eqs. (1) and (2) can be recast as the follow:

If

$$|v^{t+1}| < v_{\mathrm{b}} \tag{8}$$

then

$$d = \mathrm{e}^{-\alpha|l|} \tag{9}$$

$$v_{\mathrm{accele}} = \mathrm{sign}(v^{t+1}) \cdot \max(x_{\mathrm{SerachRange}}) \cdot d \tag{10}$$

Table 2: Numerical simulation parameters for each benchmarks

Function	Dimension	Search range	$v_{\max}$	Initial range
$f_1(x)$	10, 20, 30	(−100, 100)	100	$(50, 100)^N$
$f_2(x)$	10, 20, 30	(−100, 100)	100	$(50, 100)^N$
$f_3(x)$	10, 20, 30	(−10, 10)	10	$(2.56, 5.12)^N$
$f_4(x)$	10, 20, 30	(−600, 600)	600	$(300, 600)^N$
$f_5(x)$	2	(−100, 100)	100	$(15, 30)^N$

Table 3: Numerical simulation parameters for PSO

	w	$c_1 = c_2$	$\mathit{\Delta}$	θ[deg]	γ
PSO	0.729	1.49445	–	–	–
IWA	0.9→0.4	2.0	–	–	–
VBR	0.729	1.49445	–	–	–
D-PSO	–	–	0.95	45	0.5
CD-PSO	–	–	0.95	45	0.5

$$v^{t+1} = v^{t+1} + v_{\text{accele}} \tag{11}$$

where v_{boundary} is a lower limit of the velocity when the system reaccelerates, l denotes the distance between x_{ij}^t and $gbest_j^t$, and d denotes a reacceleration coefficient.

The reacceleration coefficient effects to improve searching ability when the velocity becomes slow. Since our proposed ”Reacceleration” procedure has a lower limit of the velocity, the system can not be a local search depends on the objective cost function and the setting of v_b.

Equation (9) corresponds to the condition that a strong acceleration is given when the difference between x_{ij}^t and $gbest_j^t$ is small. This condition inhibits the interference of the search due to the reacceleration before the velocity is reduced. Equation (10) denotes the condition that gives the maximum value of the reacceleration and its direction.

4. Simulations

In order to confirm the effects of ”Reacceleration”, we carry out numerical simulations for the standard PSO, IWA procedure, VBR procedure, the D-PSO, and the CD-PSO. For the reacceleration procedure, we set the boundary parameter as $v_\text{b} = 1\text{e} - 6$ and $v_\text{b} = 1\text{e} - 9$. We use some benchmark functions as shown in Table 1, and the parameters of each function are as shown in Table 2. Also, the parameters of PSO for numerical simulations are shown in Table 3. The numerical simulation results are shown in Table 4.

Futhermore, we confirm the effect of the rotation angle θ and the damping factor $\mathit{\Delta}$ when the acceleration operation is applied to the D-PSO and the CD-PSO. The parameters are the same as the previous experiment, however, the dimension of each benchmark function is 10 excepting schaffer's $f6$ function. As the rotation angle θ and the damping factor $\mathit{\Delta}$ are varied, Figures 1 and 2 illustrate the results the D-PSO and the CD-PSO, respectively. The lower limit of the velocity $v_\text{b} = 10^{-6}$ is applied. The color of each pixel corresponds to the evaluation value; livid color denotes good value. These figures indicate that the optimum combination value of θ and $\mathit{\Delta}$ exist for the reacceleration operation to the D-PSO and the CD-PSO.

5. Conclusions

In this article, we introduced the CD-PSO, and compared with the searching ability between the standard PSO and the CD-PSO. Also, we proposed the reacceleration operation to improve the searching performance. It can be applied to the standard PSO, the D-PSO, and the CD-PSO. The simulation results indicate that the reacceleration operation is very effective to improve the searching performance. However, the parameter tuning of the reacceleration operation is not sufficient. The theoretical analysis of the parameter is one of our future problems.

Acknowledgments

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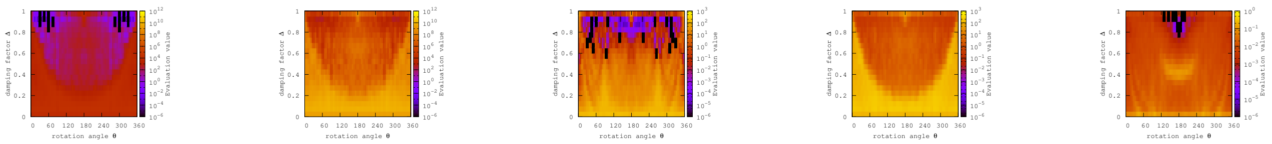
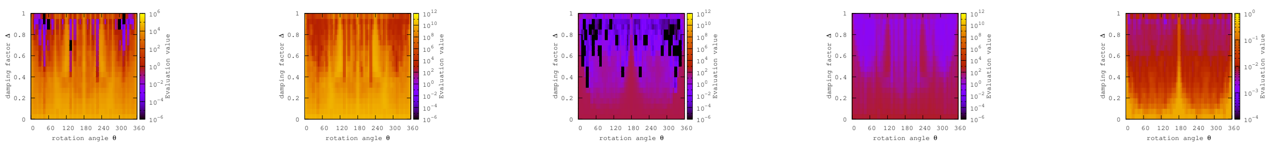
Table 4: The upper value denotes the average evaluation value of 50 trials, and the lower bracket value denotes its standard deviation. "R-PSO" means the standard PSO with reacceleration operation. For "IWA", the inertia parameter varies from 0.9 to 0.4. For "VBR", α denotes the threshold value of reinitialization. "RD-PSO" and "RCD-PSO" mean the D-PSO with reacceleration operation and the CD-PSO with reacceleration operation, respectively. The number of particles is 40, and the maximum iteration number is 5000.

Func.	Dim.	PSO	IWA $w = 0.9 \rightarrow 0.4$	VBR $\alpha = 0.001$	R-PSO $v_b = 10^{-9}$	D-PSO	RD-PSO $v_b = 10^{-6}$	RD-PSO $v_b = 10^{-9}$	CD-PSO	RCD-PSO $v_b = 10^{-6}$	RCD-PSO $v_b = 10^{-9}$
Sphere	10	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	3.91 (9.28)	0.00 (0.00)	0.00 (0.00)	13839.44 (4890.41)	0.00 (0.00)	0.00 (0.00)
	20	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	3635.09 (3059.51)	0.00 (0.00)	0.12 (0.56)	49806.52 (7536.79)	0.00 (0.00)	0.00 (0.00)
	30	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	24562.31 (12459.99)	5.85 (40.61)	19.27 (107.60)	94180.36 (10058.74)	0.00 (0.00)	0.01 (0.01)
Rosenbrock	10	0.84 (1.61)	8.36 (16.04)	0.84 (1.61)	0.52 (1.43)	1.26E+05 (6.81E+05)	123.73 (207.72)	3789.94 (25583.21)	1.93E+09 (1.47E+09)	236.73 (266.96)	2859.36 (6736.02)
	20	3.86 (11.21)	23.14 (35.09)	3.86 (11.21)	2.81 (11.48)	5.56E+08 (1.65E+09)	3376.58 (10240.71)	5021.02 (13020.09)	1.49E+10 (4.12E+09)	3192.57 (5666.04)	4523.96 (7299.55)
	30	17.47 (30.74)	69.45 (89.76)	17.47 (30.74)	13.94 (21.75)	4.96E+09 (4.56E+09)	7886.43 (13555.27)	2.98E+05 (1.62E+06)	3.58E+10 (7.16E+09)	2386.97 (4227.66)	3773.30 (4844.35)
Rastrigin	10	2.96 (4.72)	0.00 (0.00)	2.96 (4.72)	0.00 (0.00)	111.85 (22.83)	0.55 (3.86)	0.68 (3.24)	126.46 (16.95)	0.70 (2.34)	12.22 (18.92)
	20	46.95 (27.75)	0.08 (0.27)	47.09 (27.73)	0.00 (0.00)	273.19 (39.41)	5.10 (9.14)	10.37 (15.21)	296.05 (28.67)	14.44 (15.78)	90.37 (53.73)
	30	131.42 (60.31)	6.36 (4.32)	131.42 (60.31)	0.00 (0.00)	464.10 (50.39)	38.42 (23.31)	42.20 (32.72)	500.27 (39.35)	55.60 (33.18)	167.06 (84.67)
Griewank	10	0.0776 (0.0340)	0.0611 (0.0205)	0.0881 (0.0364)	0.0393 (0.0196)	0.9834 (0.8019)	0.2918 (0.1754)	0.2962 (0.1646)	125.5687 (44.2000)	0.2448 (0.1564)	0.2806 (0.1858)
	20	0.0163 (0.0159)	0.0406 (0.0353)	0.0154 (0.0164)	0.0139 (0.0151)	33.7095 (27.5025)	0.2251 (0.1606)	0.2231 (0.1806)	449.2586 (67.8310)	0.0856 (0.1365)	0.0584 (0.0841)
	30	0.0198 (0.0201)	391.6207 (42.1471)	0.0191 (0.0204)	0.0177 (0.0182)	222.0608 (112.1399)	0.7036 (2.8607)	0.9277 (3.5663)	848.6232 (90.5286)	0.0723 (0.1678)	0.0369 (0.0381)
Schaffer's f6	2	0.0019 (0.0039)	0.0000 (0.0000)	0.0019 (0.0039)	0.0010 (0.0029)	0.0509 (0.0556)	0.0041 (0.0048)	0.0056 (0.0048)	0.2430 (0.0670)	0.0012 (0.0032)	0.0038 (0.0047)



(a) Sphere function (b) Rosenbrock function (c) Rastrigin function (d) Griewank function (e) Schaffer's $f6$ function

Figure 1: The effect of the rotation angle θ and the damping factor Δ when the acceleration operation is applied to the D-PSO with $v_b = 10^{-6}$. The evaluation function consists with 10 dimensional valiable. The maximum number of iterations is 5000, and the number of trials is 50. The color of each pixel corresponds to the evaluation value; livid color denotes good value.



(a) Sphere function (b) Rosenbrock function (c) Rastrigin function (d) Griewank function (e) Schaffer's $f6$ function

Figure 2: The effect of the rotation angle θ and the damping factor Δ when the acceleration operation is applied to the CD-PSO with $v_b = 10^{-6}$. The evaluation function consists with 10 dimensional valiable. The maximum number of iterations is 5000, and the number of trials is 50. The color of each pixel corresponds to the evaluation value; livid color denotes good value.