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# Two-Dimensional Piecewise defined Maps realized with a Simple Switching Circuit

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**Abstract**—This paper deals with the model analysis of a chaos generator. In this proposed system discrete value signals and continuous value signals are used together and are interacting one another. The main interest of this system is that it allows the generation of chaotic signals from a discrete transformation by avoiding the problem of periodicity which is always encountered in any fully discrete systems. A chaotic behavior study of the system is then provided, in which bifurcation phenomena are explained and chaotic attractors are shown.
Keywords : Two-dimensional chaos generator, switched dynamical systems, piecewise map.

## 1. Introduction

Chaotic signals seem to be very useful in many applications, and specially in telecommunication. It has been shown that chaotic sequences can be used in telecommunication systems in order to improve performance of systems and also increase the security of the communications. Attractive properties of chaotic signals for this kind of utilization are under study since a long time and so, chaotic signals generators have drawn many attention.
There are many implementations of chaos generators in the literature. Some of them operate in continuous time such as systems based on the Chua's circuit (analogical circuit), while others are discrete time systems which iterate a chaotic map (digital circuit). Continuous time systems with continuous value signals can theoretically produce perfect chaotic signals (aperiodic signals). The problem with these systems is that parameters and initial conditions are very difficult to set with a great accuracy, especially in a noisy environment. Iterative chaotic maps can be used to generate chaotic signals (chaotic sequences) ; parameters and initial conditions can be exactly known, but these systems can only produce discrete values signals and then lead to periodic sequences, which can often be a problem. Some other systems are based on so-called switching circuits [1][2][3] where a latch (discrete value signal) is used to modify the behavior of a analogical circuit (continuous value signal).
Our aim is to propose a simple circuit, giving rise to periodic behaviour as well as chaotic one. We want to analyze and explain the way of obtaining chaotic signal from this circuit. In section 2, we describe the circuit and introduce the model. In section 3, bifurcations permitting to obtain periodic orbits and chaos are explained for some parameter values.

## 2. Description of the circuit

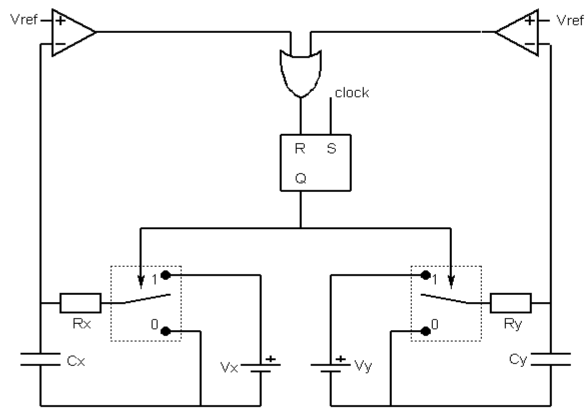


Figure 1: Schematic of the circuit

The proposed chaos generator is a quite simple circuit, very similar to those discussed in [1] and [3]. The main difference is that the proposed circuit is a two-dimensional chaos generator, i.e., with two coupled state variables. So, the proposed chaos generator in figure 1 consists of an R-S flip-flop which permits to change the position of two switches simultaneously. Depending on the position '1' or '0' of the switches, two capacitor-resistance circuits $C_x$-$R_x$ and $C_y$-$R_y$ are supplied with DC voltage sources $V_x$ and $V_y$ respectively, or connected to the ground. State variables of the system are the two voltages capacitor $v_x(t)$ and $v_y(t)$. At every clock period $T$, the flip-flop is set and then the switches position is '1'. When one of the capacitance voltages reaches the reference value $V_{ref}$, the two switches are turned toward their position '0'. So, according to the switches position, the two capacitors are simultaneously charging or discharging.

At the $n^{th}$ rising edge of the clock, the two state variables are given by $v_x(nT)$ and $v_y(nT)$, and the capacitors start charging. Then, it requires the duration $t_x^{(n)} = R_xC_x \ln\left(\frac{V_x - v_x(nT)}{V_x - V_{ref}}\right)$ to the voltage capacitor $v_x(t)$ to reach $V_{ref}$ and, similarly it requires the duration $t_y^{(n)} = R_yC_y \ln\left(\frac{V_y - v_y(nT)}{V_y - V_{ref}}\right)$ to the voltage capacitor $v_x(t)$ to reach $V_{ref}$. Then three cases can happen :

1. none of the two states variables have reached the reference voltage $V_{ref}$ before the next rising edge of the clock. This case appears when $t_x^{(n)} < T$ and $t_y^{(n)} < T$, or equivalently when $v_x(nT) < V_x - (V_x - V_{ref})e^{\frac{T}{R_xC_x}}$ and $v_y(nT) < V_y - (V_y - V_{ref})e^{\frac{T}{R_yC_y}}$. Then no switch has occurred and values of the state variables at the $(n+1)^{th}$ rising edge are :

$$\begin{cases} v_x((n+1)T) = V_x + (v_x(nT) - V_x)e^{-\frac{T}{R_xC_x}} \\ v_y((n+1)T) = V_y + (v_y(nT) - V_y)e^{-\frac{T}{R_yC_y}} \end{cases} \tag{1}$$

2. voltage capacitor $v_x(t)$ reaches the reference voltage $V_{ref}$ before $v_y(t)$ can do it and before the next rising edge. It happens when $v_x(nT) \geq V_x - (V_x - V_{ref})e^{\frac{T}{R_xC_x}}$ and when $t_x^{(n)} \leq t_y^{(n)}$. Then the two switches are turned to '0' at time $nT + t_x^{(n)}$ and values of the state variables at the $(n+1)^{th}$ rising edge are :

$$\begin{cases} v_x((n+1)T) = V_{ref}\frac{V_x - v_x(nT)}{V_x - V_{ref}}e^{-\frac{T}{R_xC_x}} \\ v_y((n+1)T) = \\ \left(V_y\left(\frac{V_x - v_x(nT)}{V_x - V_{ref}}\right)^{\frac{R_xC_x}{R_yC_y}} - V_y + v_y(nT)\right)e^{-\frac{T}{R_yC_y}} \end{cases} \tag{2}$$

3. voltage capacitor $v_y(t)$ reaches the reference voltage $V_{ref}$ before the next rising edge. This case is the analogous of the previous one. It happens when $v_y(nT) \geq V_y - (V_y - V_{ref})e^{\frac{T}{R_yC_y}}$ and when $t_x^{(n)} \geq t_y^{(n)}$. Then the two switches are turned to '0' at time $nT + t_y^{(n)}$ and values of the state variables at the $(n+1)^{th}$ rising edge are :

$$\begin{cases} v_x((n+1)T) = \\ \left(V_x\left(\frac{V_y - v_y(nT)}{V_y - V_{ref}}\right)^{\frac{R_yC_y}{R_xC_x}} - V_x + v_x(nT)\right)e^{-\frac{T}{R_xC_x}} \\ v_y((n+1)T) = V_{ref}\frac{V_y - v_y(nT)}{V_y - V_{ref}}e^{-\frac{T}{R_yC_y}} \end{cases} \tag{3}$$

For simplicity in the rest of this work only a special configuration of this circuit will be studied, when $V_x = V_y$. Let now define the normalized parameters of the model :

$$\begin{array}{ll} \alpha = \frac{V_x}{V_{ref}} & \text{with } \alpha > 1 \\ \mu = \frac{R_yC_y}{R_xC_x} & \text{with } \mu > 0 \\ \delta = e^{-\frac{T}{R_xC_x}} & \text{with } 0 < \delta < 1 \end{array} \tag{4}$$

Normalized state variables are given such that :

$$\begin{array}{ll} x_n = \frac{v_x(nT)}{V_{ref}} & \text{with } 0 \leq x_n \leq 1 \\ y_n = \frac{v_y(nT)}{V_{ref}} & \text{with } 0 \leq y_n \leq 1 \end{array} \tag{5}$$

Let us define the following curves in $[0,1]^2$:

$$\begin{array}{l} LC_{-1}^1 : x_n = x_b \\ LC_{-1}^2 : y_n = y_b \\ LC_{-1}^3 : \Delta(x_n, y_n) = 0 \end{array} \tag{6}$$

where $x_b = \alpha + \frac{1-\alpha}{\delta}$, $y_b = \alpha + \frac{1-\alpha}{\delta^{1/\mu}}$, $\Delta(x_n, y_n) = \left(\frac{y_n - \alpha}{1-\alpha}\right)^{\mu} - \frac{x_n - \alpha}{1-\alpha}$, and the following domains in $[0,1]^2$:

$$\begin{array}{l} D_1 = \{(x_n, y_n)/x_n \leq x_b \text{ and } y_n \leq y_b\} \\ D_2 = \{(x_n, y_n)/x_n > x_b \text{ and } \Delta(x_n, y_n) > 0\} \\ D_3 = \{(x_n, y_n)/y_n > y_b \text{ and } \Delta(x_n, y_n) \leq 0\}. \end{array} \tag{7}$$

Then the circuit model is given by the map $T$, which is defined as follows :

$$\begin{array}{ll} \text{if } (x_n, y_n) \in D_1, T(x_n, y_n) = & \begin{cases} \alpha + (x_n - \alpha)\delta \\ \alpha + (y_n - \alpha)\delta^{1/\mu} \end{cases} \\ \text{if } (x_n, y_n) \in D_2, T(x_n, y_n) = & \begin{cases} \frac{x_n - \alpha}{1-\alpha}\delta \\ \left(\alpha\left(\frac{x_n - \alpha}{1-\alpha}\right)^{1/\mu} - \alpha + y_n\right)\delta^{1/\mu} \end{cases} \\ \text{if } (x_n, y_n) \in D_3, T(x_n, y_n) = & \begin{cases} \left(\alpha\left(\frac{y_n - \alpha}{1-\alpha}\right)^{\mu} - \alpha + x_n\right)\delta \\ \frac{y_n - \alpha}{1-\alpha}\delta^{1/\mu} \end{cases} \end{array} \tag{8}$$

We shall denote $T_{|D_1}$ by $T_1$ (respectively $T_{|D_2}$ by $T_2$ and $T_{|D_3}$ by $T_3$. In the phase plane $(x_n, y_n)$, the curves $LC_{-1}^1$, $LC_{-1}^2$ and $LC_{-1}^3$ define the switching lines and are also called critical curves (see figure 2).

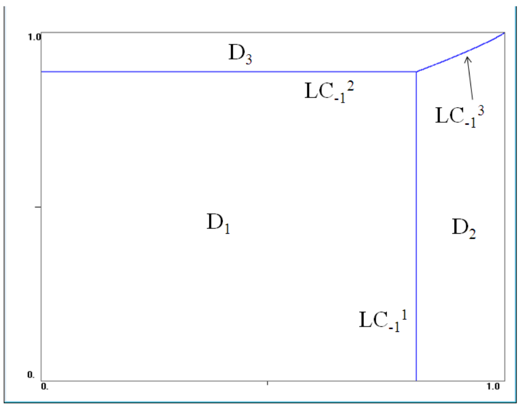


Figure 2: In the phase plane $(x_n, y_n)$, switching lines $LC_{-1}^i, i = 1,2,3$ and domains $D_i, i = 1,2,3$.

## 3. Analysis of bifurcations and route to chaos

In this section, we propose to analyse bifurcations occurring in the circuit modeled by (8). The model is given by a continuous piecewise smooth map, so we know that the critical curves $LC_{-1}^{i}, i = 1,2,3$ and their images by $T$ permit to explain some of the bifurcations that occur [5]. We have chosen to fix $\alpha$ and to study the parameter plane $(\mu, \delta)$. The figure 3 is obtained by simulations in the parameter plane $(\mu, \delta)$ when $\alpha = 1.1$ and the figure 4 when $\alpha = 1.01$. Each colored part corresponds to the existence of at least a periodic orbit, which order $k$ corresponds to the indicated colour on the figure (blue : $k$=1, red : $k$=2,...). The black colour corresponds to order greater than 14 or to chaotic attractors. We propose to explain bifurcations by considering first the fixed points, then the evolution of periodic orbits of order greater than 1 when $\mu < 1$ and $\mu > 1$.

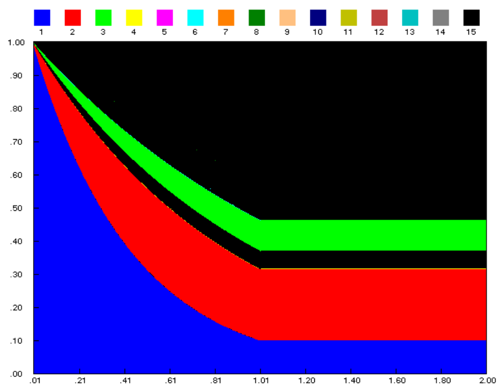


Figure 3: Parameter plane $(\mu, \delta)$, $\alpha = 1.1$. Only attractive periodic orbit of order 1 and 3 exist.

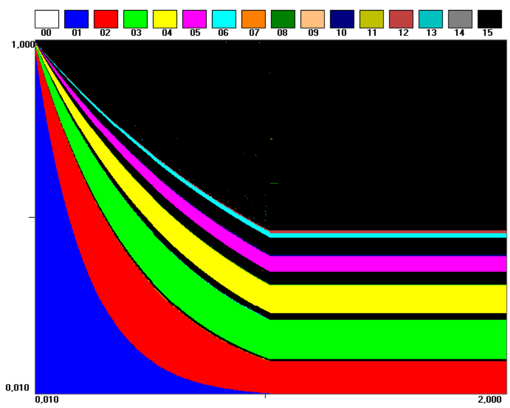


Figure 4: Parameter plane $(\mu, \delta)$, $\alpha = 1.01$. Attractive periodic orbits of order 1 to 6 exist.

### 3.1. Study of fixed points

To find the fixed points, we have to determine the fixed points of each determination of $T$ (i.e. $T_1$, $T_2$, $T_3$) and to verify if they belong or not to the domain on which each $T_i, i = 1,2,3$ is defined.

- The fixed point of $T_1$ is denoted by $X_1(x_1 = \alpha, y_1 = \alpha)$. This point does not belong to $D_1$. It is called a virtual fixed point [5].

- $\mu > 1$
  The fixed point of $T_2$ is denoted by $X_2(x_2, y_2)$:

$$x_2 = \frac{\alpha\delta}{\alpha+\delta-1}, \quad y_2 = \left(\left(\frac{\alpha}{\alpha+\delta-1}\right)^{\frac{1}{\mu}} - 1\right)\frac{\alpha\delta^{\frac{1}{\mu}}}{1-\delta^{\frac{1}{\mu}}} \tag{9}$$

  $X_2$ belongs to $D_2$ when $\mu > 1$. The eigenvalues of the Jacobean matrix of $T_2$ are :

$$\lambda_1 = \frac{\delta}{1-\alpha}, \quad \lambda_2 = \delta^{\frac{1}{\mu}} \tag{10}$$

  It is easy to check that $\lambda_1 < 0$ and $\lambda_2 \in ]0,1[$. A flip bifurcation occurs for $X_2$ when $\delta = \alpha - 1$. This curve corresponds to the straight line between the blue and red part in the parameter plane $(\mu, \delta)$ (Figures 3 and 4). $\delta$ is less than 1, so it implies that $\alpha$ must be less than 2 in order to obtain a flip bifurcation for the fixed point $X_2$. An attractive order 2 cycle exists when $\alpha - 1 < \delta$ until it becomes repulsive by undergoing a flip bifurcation.

- $\mu < 1$
  The fixed point of $T_3$ is denoted by $X_3(x_3, y_3)$:

$$x_3 = \frac{\alpha\delta}{1-\delta}\left(\left(\frac{\alpha}{\alpha+\delta^{\frac{1}{\mu}}-1}\right)^{\mu} - 1\right), \quad y_3 = \frac{\alpha\delta^{\frac{1}{\mu}}}{\alpha+\delta^{\frac{1}{\mu}}-1} \tag{11}$$

  $X_3$ belongs to $D_3$ when $\mu < 1$. The eigenvalues of the Jacobean matrix of $T_3$ are :

$$\lambda_1 = \delta, \quad \lambda_2 = \frac{\delta^{\frac{1}{\mu}}}{1-\alpha} \tag{12}$$

  It is easy to check that $\lambda_2 < 0$ and $\lambda_1 \in ]0,1[$. A flip bifurcation occurs for $X_3$ when $\delta^{\frac{1}{\mu}} = \alpha - 1$. The same remarks hold for the fixed point $X_3$ than for $X_2$ : the flip bifurcation can be obtained only if $\alpha < 2$. An attractive order 2 cycle exists when $\alpha - 1 < \delta^{\frac{1}{\mu}}$ until it undergoes itself a flip bifurcation by border-collision.

Let consider now some of the bifurcations occurring for periodic orbits of order greater than 2. This part is under study and will be developed in future works.

### 3.2. Case : $\mu > 1$

By considering analytical and numerical studies, we can conjecture the following properties :

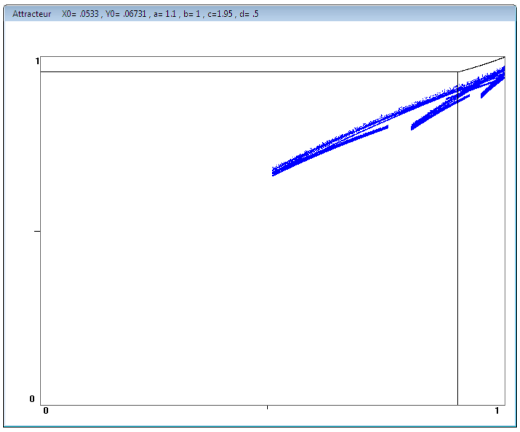


Figure 5: Chaotic attractor in the $(x_n, y_n)$ plane $\alpha = 1.1$, $\delta = 0.5$, $\mu = 1.95$. The chaotic attractor is located in the domains $D_1$ and $D_2$.

- The points of an order $k$ cycle belong to $D_1$ and $D_2$, so the points are exchanged by $T_1$ and $T_2$, the map $T_3$ is not involved in this case. $k-1$ points belong to $D_1$, one belongs to $D_2$. One point of an order $k$ cycle is a fixed point of the map $T_1^{k-1}T_2$.
- The study of the eigenvalues of the Jacobean matrix of $T_1^{k-1}T_2$ permits to show that an order $k$ cycle undergoes a flip bifurcation when $\delta^k = \alpha - 1$.
- The order $k$ cycles appear when one point is crossing a switching curve $LC_{-1}^i$, $i = 1, 2, 3$. This kind of bifurcation is called a border collision. Chaotic attractors also appear by border collision [5]. See figure 5 for a single chaotic attractor and figure 6 for an order 6 cyclic chaotic attractor.

### 3.3. Case : $\mu < 1$

The same properties occur, but regarding the maps $T_1$ and $T_3$ instead of $T_1$ and $T_2$. In this case, the flip bifurcations of order $k$ cycles are obtained when $\delta^{\frac{k}{\mu}} = \alpha - 1$. Chaotic attractors also appear by border collision.

## 4. Conclusion

We have proposed a circuit giving rise to chaotic behaviour and a model by the way of a two-dimensional piecewise smooth map. We can determine the parameters giving rise to periodic or chaotic behavior. The model depends upon three parameters ; we have fixed the parameter $\alpha$ and studied the parameter plane $(\mu, \delta)$. It is noteworthy to remark that chaos can appear in the circuit when the parameter $\alpha$ is slightly greater than 1, when the voltage source is slightly greater than the voltage reference ; if $\alpha$ becomes greater than 2, only fixed points exist. The circuit

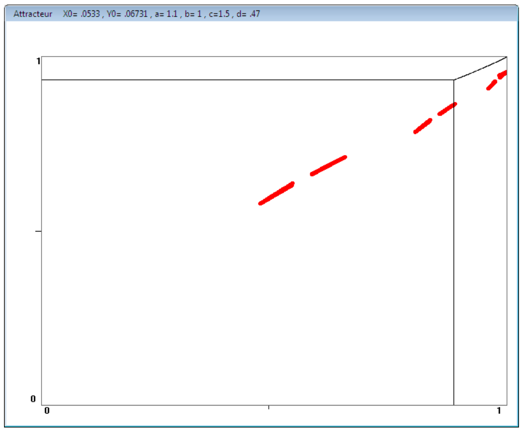


Figure 6: Order 6 cyclic chaotic attractor in the $(x_n, y_n)$ plane, $\alpha = 1.1$, $\delta = 0.47$, $\mu = 1.5$.

that we have proposed could be used as a chaotic generator in different kinds of applications (secure transmission,...).


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