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Investigation of Frustration in Several Types of Coupled Polygonal Oscillatory Networks

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Abstract—In this study, we consider the several types of circuit models which are that odd and even polygonal oscillatory networks are coupled by sharing a branch. We apply the theoretical analysis by using the whole power consumption to solve the phase difference between the adjacent oscillators. Furthermore, we discuss the general network model which is a number of polygonal oscillatory networks are coupled by multiple branches.

1. Introduction

Coupled oscillatory systems are good models to express essential role of high-dimensional nonlinear phenomena occurring in the field of natural sciences. Endo et al. have presented the details of a theoretical analysis and corresponding circuit experiments on electrical circuits oscillators arranged in a ladder, a ring and in a two-dimensional array topology [1]-[3]. Moreover, coupled oscillatory systems can also produce interesting phase patterns, including wave propagation, clustering and complex patterns [4].

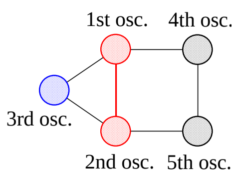
On the other hand, there are several types of polygonal network structures (e.g. Honeycomb structure and crystal structure) in the natural science. Generally, for the studies of large-scale network using coupled oscillators, a ring, a ladder and a two dimensional array structure are often investigated. However, there are not many discussions about coupled polygonal oscillatory networks by using electrical oscillators.

In our previous study, synchronization phenomena in two coupled polygonal oscillatory networks with frustration was investigated. In this system, odd number of van der Pol oscillators are connected to every corner of polygonal network and frustration is occurred by the shared branch. We have confirmed that the phase difference between the shared oscillators was shifted, then other oscillators synchronized to compensate this phase shift. In order to solve the phase difference in the circuit system, we focused on the power consumption of the coupling resistors in the whole system and proposed the theoretical analysis method by finding the minimum value of the power consumption function. By using computer simulations and theoretical analysis, we confirmed that coupled oscillators tended to synchronize to minimize the power consumption of the whole system [5].

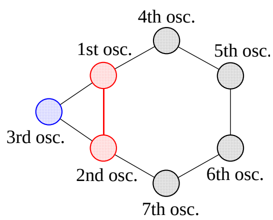
In this study, we consider the circuit models which are that odd and even polygonal oscillatory networks are coupled by sharing a branch. We apply the proposed theoretical analysis by using the whole power consumption to solve the phase difference between the adjacent oscillators. Furthermore, we discuss the general network model which is a number of polygonal oscillatory networks are coupled by multiple branches. We expect that the results of this study contribute to understanding of synchronization phenomena observed in general complex networks.

2. Coupled Odd-Even Polygonal Oscillatory Networks

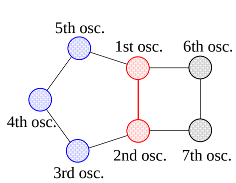
First, we investigate the synchronization phenomena in coupled odd-even polygonal oscillatory systems. We consider the four types of circuit systems as shown in Fig. 1. Two polygonal oscillatory networks are coupled by sharing a branch. We call the first and the second oscillators which are connected to both side of polygonal network "shared oscillators."



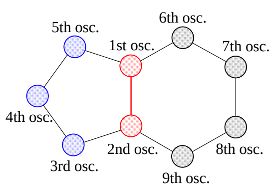
(a) 3-4 coupling network.



(b) 3-6 coupling network.



(c) 5-4 coupling network.



(d) 5-6 coupling network.

Figure 1: Two coupled odd-even number polygonal oscillatory networks.

Next, we develop the expression for the circuit equations of $3-4$ coupling oscillatory networks as shown in Fig. 2. The $v_k - i_{Rk}$ characteristics of the nonlinear resistor are ap-

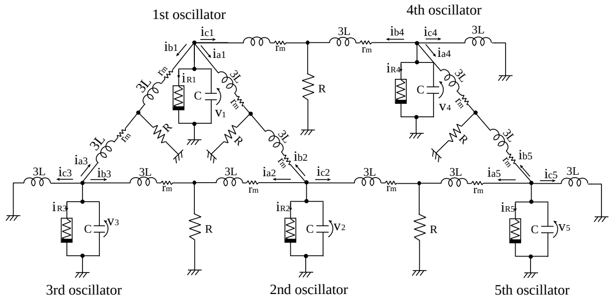


Figure 2: Circuit model for $3-4$ coupling oscillatory networks.

proximated by the following third order polynomial equation,

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3 \quad (g_1, g_3 > 0), (k = 1, 2, 3, 4, 5). \quad (1)$$

The normalized circuit equations governing the circuit are expressed as
[kth oscillator]

$$\begin{cases} \dfrac{dx_k}{d\tau} = \varepsilon\left(1 - \dfrac{1}{3}{x_k}^2\right)x_k - (y_{ak} + y_{bk} + y_{ck}) \\ \dfrac{dy_{ak}}{d\tau} = \dfrac{1}{3}\left\{x_k - \eta y_{ak} - \gamma(y_{ak} + y_n)\right\} \\ \dfrac{dy_{bk}}{d\tau} = \dfrac{1}{3}\left\{x_k - \eta y_{bk} - \gamma(y_{bk} + y_n)\right\} \\ \dfrac{dy_{ck}}{d\tau} = \dfrac{1}{3}\left\{x_k - \eta y_{ck} - \gamma(y_{ck} + y_n)\right\} \\ \qquad\qquad\qquad\qquad (k = 1, 2, 3, 4, 5). \end{cases} \quad (2)$$

In this equations, γ is the coupling strength, ε denotes the nonlinearity of the oscillators and y_n denotes the current of neighbor oscillator on coupling resistor. For the computer simulations, we calculate Eq. (2) using a fourth-order Runge-Kutta method with the step size $h = 0.005$. The parameters of this circuit model are fixed as $\varepsilon = 0.1$, $\gamma = 0.1$, $\eta = 0.001$.

2.1. Synchronization Phenomena

We investigate phase difference between adjacent oscillators in two coupled polygonal oscillators for four types of circuit model. The simulation results are summarized in Table 1. There are three phase difference types as follows; phase difference between shared oscillators (θ), phase difference of odd polygonal oscillatory network (φ_{N_o}), phase difference of even polygonal oscillatory network (φ_{N_e}). Where, N_o and N_e denote the number of coupled van der Pol oscillators to odd and even polygonal network, respectively.

From this table, we can see that the coupled oscillators do not synchronize with N_o-phase state or anti phase state in each case. In the case of that the two coupled polygonal oscillatory networks are not coupled, N_o-phase synchronization could be occurred in the odd polygonal network and anti-phase synchronization could be obtained in the even polygonal network. By adding some sort of frustration (effect of the sharing branch), the different types of synchronization can be observed.

Table 1: Phase difference of oscillatory networks (computer simulation)

Circuit Model	Phase difference		
Odd-Even N_o-N_e	Shared osc. θ	Odd osc. φ_{N_o}	Even osc. φ_{N_e}
3-4	138.60 °	109.50°	166.70°
3-6	133.30°	112.23°	171.11°
5-4	152.70°	141.00°	171.00°
5-6	150.31°	142.56°	174.15°

As one example, Fig. 3 shows the time wave forms of the voltage charged at the capacitance of each oscillator and Lissajous figures obtained from $3-4$ coupling network by using the computer simulation.

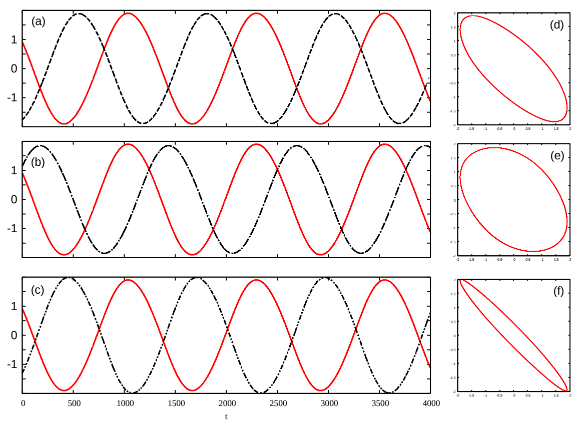


Figure 3: Time wave forms and Lissajous figures observed in $3-4$ coupling network. (a) Time wave forms of 1st (red) and 2nd (black) oscillators. (b) Time wave forms of 1st (red) and 3rd (black) oscillators. (c) Time wave forms of 1st (red) and 4th (black) oscillators. (d) Lissagous figure with 1st and 2nd oscillators. (e) Lissagous figure with 1st and 3rd oscillators. (f) Lissagous figure with 1st and 4th oscillators.

2.2. Theoretical Analysis

In this section, we apply the proposed theoretical analysis [5] to solve the phase difference of the shared oscillators and the other combination oscillators.

We assume three points as following. First, the phase difference between shared oscillators is θ. Second, the phase difference of the coupled oscillators in odd polygonal network is described by Eq. (3).

$$\varphi_{N_o} = \pi - \frac{\theta}{N_o - 1}. \quad (3)$$

Third, the phase difference of the coupled oscillators in even polygonal network is described by Eq. (4).

$$\varphi_{N_e} = \pi - \frac{\pi - \theta}{N_e - 1}. \tag{4}$$

In the case of $N_o - N_e$ coupling system, the power consumption of the whole system is expressed by following equation.

$$\begin{aligned} P &= \frac{1}{2\pi}\int_0^{2\pi}\{\sin\omega t + \sin(\omega t+\theta)\}^2 dt \\ &+ \frac{N_o-1}{2\pi}\int_0^{2\pi}\Big\{\sin\omega t \\ &+ \sin\Big(\omega t + \pi - \frac{\theta}{N_o-1}\Big)\Big\}^2 dt \\ &+ \frac{N_e-1}{2\pi}\int_0^{2\pi}\Big\{\sin\omega t \\ &+ \sin\Big(\omega t + \pi - \frac{\pi-\theta}{N_e-1}\Big)\Big\}^2 dt, \end{aligned} \tag{5}$$

where the amplitute of current is set to 1 and the coupling resistance is fixed with $R = 1$.

$$\begin{aligned} P &= 1+\cos\theta + (No-1)\{1+\cos(\pi-\theta/(No-1))\} \\ &+ (Ne-1)\{1+\cos(\pi-(\pi-\theta)/(Ne-1))\}. \end{aligned} \tag{6}$$

The extreme value is solved by Eq. (8).

$$\frac{dP}{d\theta} = -\sin\theta + \sin\Big(\pi - \frac{\theta}{N_o-1}\Big) + \sin\Big(\pi - \frac{\pi-\theta}{N_e-1}\Big). \tag{7}$$

When $\frac{dP}{d\theta} = 0$ is satisfied, θ is phase difference of the shared oscillators to minimize the power consumptions of the whole system. Table 2 summarizes the result of numerically-calculated phase difference by using Eq. (8). We confirm that the value of phase difference obtained from the theoretical analysis has similar value with the computer simulation results (see. Table 1).

Table 2: Phase difference of oscillatory networks (theoretical analysis)

Circuit Model	Phase difference		
Odd-Even	Shared osc.	Odd osc.	Even osc.
3-4	137.42 °	111.29°	165.81°
3-6	131.69°	114.16°	170.34°
5-4	152.32°	141.92°	170.77°
5-6	150.02°	142.50°	174.00°

As an example, Fig. 4 shows the graphs of Eqs. (6), (7) for 3-6 coupling network. From this figure, we can see that extreme value of Eq. (6) corresponds to 0 (zero) of Eq. (7).

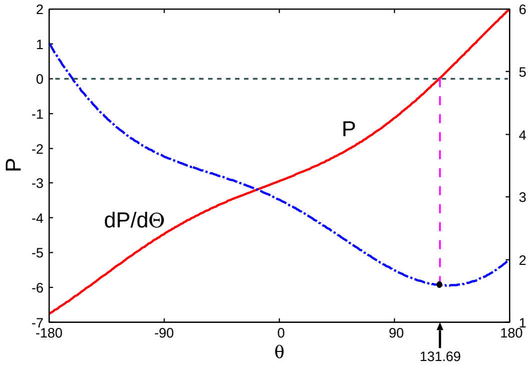


Figure 4: Function of power consumption for 3-6 coupling network.

3. Application for General Networks

We apply the previous theoretical approach to solve the phase difference of shared branch to the general networks. One example of general coupled polygonal network is shown in Fig. 5.

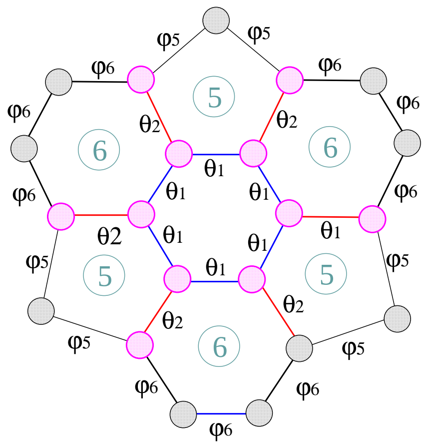


Figure 5: One example of general coupled oscillatory networks.

Then we can obtain the phase differences φ_5 and φ_7 as described by the following equation.

$$\varphi_5 = \frac{4\pi - (\theta_1 + 2\theta_2)}{2}. \tag{8}$$

$$\varphi_6 = \pi - \frac{(\pi-\theta_1) + 2(\pi-\theta_2)}{3}. \tag{9}$$

The power consumption equation is expressed as

$$\begin{aligned} P &= 6(1+\cos\theta_1) + 6(1+\cos\theta_2) \\ &+ 6\Big(1+\cos\frac{4\pi-(\theta_1+2\theta_2)}{2}\Big) \\ &+ 12\Big(1+\cos\frac{(\pi-\theta_1)+2(\pi-\theta_2)}{3}\Big). \end{aligned} \tag{10}$$

Figure 6 shows the function of power consumption with 3D plot when θ_1 and θ_2 are changed from 0 to π. By calculating this function, we obtain θ_1 and θ_2 which show the minimum value of this function. In the range of [0:π], the minimum value of θ_1 and θ_2 are 167.30° and 152.41°, respectively.

Table 3 summarizes the phase differences obtained from the computer simulations and the theoretical analysis using Eqs. (8)-(10). We confirm that the phase differences of the theoretical analysis match pretty well to the results of the computer simulations. We consider that even if calculation becomes more complex for the asymmetrical system, the same theoretical analysis can be applied.

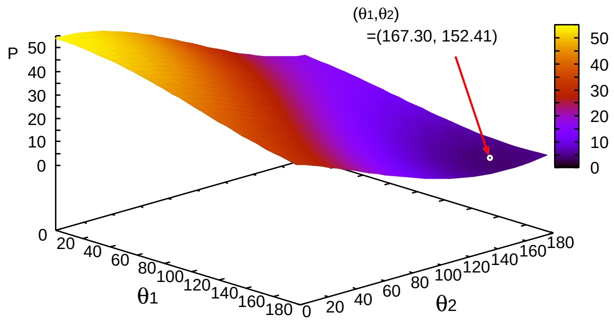


Figure 6: One example of general coupled oscillatory networks.

Table 3: Phase difference for general oscillatory networks

Phase Type	Phase Difference	
	Simulation	Theory
θ_1	167.68°	167.30°
θ_2	153.85°	152.41°
φ_5	122.78°	123.94°
φ_6	158.40°	157.37°

4. Conclusion

In this study, we have investigated synchronization phenomena in coupled polygonal oscillatory networks with frustration. By using computer simulations and theoretical analysis, we have confirmed that coupled oscillators tend to synchronize to minimize the power consumption of whole system. Investigation of synchronization phenomena observed in coupled chaotic oscillator in the polygonal networks with frustration is our future work.

Acknowledgment

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