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Formulating Hybrid Equations for Nonlinear Circuits Using SPICE

Mitsuru Tonokura and Kiyotaka Yamamura
Faculty of Science and Engineering, Chuo University
Tokyo, 112-8551 Japan
Email: yamamura@elect.chuo-u.ac.jp

***Abstract*— Hybrid equations are often used in the theoretical study of nonlinear resistive circuits because they have an easily analyzed structure. They are also advantageous in the numerical analysis of nonlinear resistive circuits because they consist of a small number of variables and are separable. However, the hybrid equations are seldom used in practical application because their formulation is difficult. In this paper, we propose a simple method for formulating the hybrid equations using SPICE. In the proposed method, we only perform the transient analysis of SPICE to a linear circuit that is obtained through a small modification to the original circuit. It is also shown that state equations of nonlinear dynamic circuits can also be formulated by the proposed method.**

I. Introduction

There are several types of equations that describe nonlinear resistive circuits. Among them, hybrid equations [1, p.291] are often used in the theoretical study of nonlinear resistive circuits (such as the existence, uniqueness, and stability of solutions [1]–[5] or proving the global convergence of homotopy methods [6]) because they have an easily analyzed structure. Hybrid equations are also advantageous in the numerical analysis of nonlinear resistive circuits (such as piecewise-linear analysis [7],[8] and finding all solutions [9]–[12]) because they consist of a small number of variables and are separable.

However, the hybrid equations are seldom used in practical applications because their formulation is difficult [1]. In circuit simulation, modified nodal equations are widely used because they can be easily formulated. In the circuit simulator SPICE, the modified nodal equations are automatically formulated from the netlists. Hence, designers need not formulate them by hand or by their own programs. Conversely, if we can easily formulate the hybrid equations "using SPICE only," "without complicated theory and programming," then the hybrid equations may be widely used as practical equations.

In this paper, we propose a simple method for formulating the hybrid equations using SPICE. In the proposed method, we only perform the transient analysis of SPICE to a linear circuit that is obtained through a small modification to the original circuit. Hence, complicated theory and programming are not necessary. Note that the proposed method is a kind of the SPICE-oriented approach [13]. In this paper, it is also shown that state equations of nonlinear dynamic circuits can also be formulated by the proposed method.

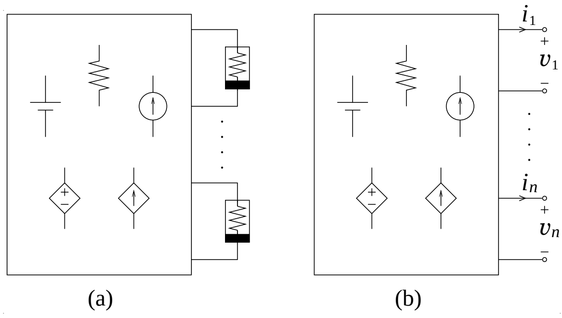

Fig. 1. Extraction of nonlinear resistors to form a linear n-port.

II. Hybrid Equation

In this section, we first review the hybrid equations briefly. For details, see [1] and [3].

Consider a nonlinear resistive circuit N that consists of n nonlinear resistors (coupled to each other or uncoupled), linear resistors, constant independent sources, and linear controlled sources. Let $\boldsymbol{v}_a$ and $\boldsymbol{i}_a$ be the vectors of branch voltages and currents, respectively, of the voltage-controlled nonlinear resistors, and let $\boldsymbol{v}_b$ and $\boldsymbol{i}_b$ be the vectors of branch voltages and currents, respectively, of the current-controlled nonlinear resistors. As shown in Fig. 1(a), we first extract all nonlinear resistors and replacing them by ports. Then, the remaining n-port as shown in Fig. 1(b) (that contains only linear resistors, linear controlled sources, and independent sources) is described by a hybrid n-port representation of the form:

$$\begin{bmatrix} \boldsymbol{i}_a \\ \boldsymbol{v}_b \end{bmatrix} + \boldsymbol{H} \begin{bmatrix} \boldsymbol{v}_a \\ \boldsymbol{i}_b \end{bmatrix} - \boldsymbol{s} = \boldsymbol{0}, \qquad (1)$$

where $\boldsymbol{H}$ is an $n \times n$ matrix called the hybrid matrix and $\boldsymbol{s}$ is an n-dimensional vector accounting for the independent sources. Such a hybrid representation exists if, and only if, the voltage ports, together with the internal voltage sources do not form any loops, and the current ports, together with the internal current sources do not form any cut-sets [1],[3]. It has been shown that there is little loss of generality in this assumption [1], and a large class of nonlinear resistive circuits has the hybrid representation of the form (1).

Let the constitutive relations of the nonlinear resistors across the ports be represented by $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$ where $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_a \\ \boldsymbol{i}_b \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{i}_a \\ \boldsymbol{v}_b \end{bmatrix}$. Combining this equation with the above hybrid

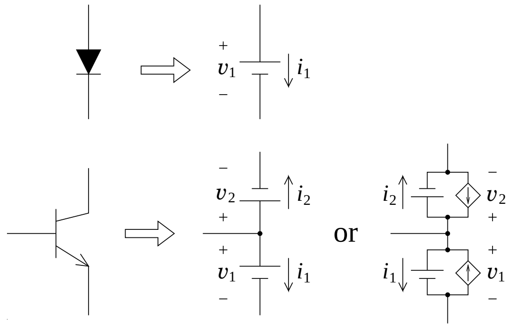

Fig. 2. Replacement of nonlinear resistors by independent voltage sources.

representation, we obtain the hybrid equation:

$$\boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{H}\boldsymbol{x} - \boldsymbol{s} = \boldsymbol{0}. \tag{2}$$

Thus, the hybrid equations are equations whose variables are branch voltages and/or currents of nonlinear resistors.

However, formulating the hybrid equations is quite involved as shown in Chapter 6 of [1]; it requires selecting a tree that satisfies some conditions, producing a large number of equations, and eliminating unnecessary variables by matrix operations.

III. Formulating Hybrid Equations Using SPICE

In this section, we first propose a simple method for formulating the hybrid equations using the DC analysis of SPICE. Then, we propose a more efficient method using the transient analysis of SPICE. For the simplicity of discussion, in this section we assume that all nonlinear resistors are voltage-controlled. We will add a simple explanation in the case that nonlinear resistors are current-controlled at the end of this section.

A. Method Using DC Analysis

Assume that the nonlinear resistive circuit N can be described by a hybrid equation of the form (2). We first replace each nonlinear resistor in N by an independent voltage source as shown in Fig. 2 and consider the linear circuit $\hat{N}$ thus obtained. We will call these independent voltage sources as *replaced* voltage sources. For a bipolar transistor, there are two ways of this replacement; one is replacing the transistor by two independent voltage sources, and the other is modeling the transistor by the Ebers-Moll model and then replacing the diodes in the Ebers-Moll model by independent voltage sources (see Fig. 2). The resulting hybrid equations are of course different [3, p.41]. The former way is often used when the transistors are modeled by non-separable models such as the Gummel-Poon model. Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^T$ and $\boldsymbol{i} = (i_1, i_2, \cdots, i_n)^T$ be the vectors of the branch voltages and currents, respectively, of the replaced voltage sources. Then, (1) becomes

$$\boldsymbol{i} + \boldsymbol{H}\boldsymbol{v} - \boldsymbol{s} = \boldsymbol{0}. \tag{3}$$

Now we set all the replaced voltage sources to 0V. Then, substituting $\boldsymbol{v} = \boldsymbol{0}$ in (3), we obtain

$$\boldsymbol{i} = \boldsymbol{s}. \tag{4}$$

Hence, the vector $\boldsymbol{s}$ is obtained from the branch currents of the replaced voltage sources. Such a circuit where nonlinear resistors of the original circuit are replaced by independent voltage sources with value zero will be called the $\boldsymbol{s}$-circuit.

Consider again the linear circuit $\hat{N}$ where n nonlinear resistors of the original circuit N are replaced by independent voltage sources. We first set "all the independent sources that were contained in N" to 0. Then, $\boldsymbol{s} = \boldsymbol{0}$ holds. We next set only the jth voltage source of the replaced voltage sources to -1V, and set the other replaced voltage sources to 0V. Then, $\boldsymbol{v} = (0, \cdots, -1_j, \cdots, 0)^T$ holds. Such a circuit will be called the $\boldsymbol{h}_j$-circuit. From (3), we have

$$\boldsymbol{i} = \boldsymbol{h}_j, \tag{5}$$

where $\boldsymbol{h}_j$ denotes the jth column vector of the matrix $\boldsymbol{H}$ [namely, $\boldsymbol{H} = (\boldsymbol{h}_1, \cdots, \boldsymbol{h}_j, \cdots, \boldsymbol{h}_n)$]. Hence, the vector $\boldsymbol{h}_j$ is obtained from the branch currents of the replaced voltage sources of the $\boldsymbol{h}_j$-circuit.

Thus, we can obtain the vector $\boldsymbol{s}$ and the matrix $\boldsymbol{H}$ by performing the DC analysis of SPICE to the $\boldsymbol{s}$-circuit and the $\boldsymbol{h}_j$-circuits $(j = 1, 2, \cdots, n)$. In SPICE, branch currents of voltage sources are automatically defined as variables, and we can obtain their values from the results of DC analysis.

This is the method using DC analysis of SPICE. Considering the structure of (2), this method is almost trivial.

B. Method Using Transient Analysis

In the previous method, we have to perform the DC analysis to the $\boldsymbol{s}$-circuit and the $\boldsymbol{h}_j$-circuits $(j = 1, 2, \cdots, n)$. However, performing the DC analysis $n+1$ times is a troublesome task when n is large. In this section, we propose a more efficient method using the transient analysis of SPICE.

We first replace each independent source that was contained in N by a pulse source that takes the same value as that of the independent source on the time interval $[0, 1]$ and 0 on $[1, n+1]$. Then, replace each nonlinear resistor by a pulse voltage source where the jth pulse voltage source takes the value -1V on $[j, j+1]$ and 0V on other time intervals. Such a linear circuit will be denoted by $\hat{N}_p$. If we perform the transient analysis to $\hat{N}_p$, then virtually DC analysis to the $\boldsymbol{s}$-circuit is performed on the time interval $[0, 1]$ and DC analysis to the $\boldsymbol{h}_j$-circuit is performed on $[j, j+1]$ $(j = 1, 2, \cdots, n)$. Hence, we can obtain the hybrid equations from the waveforms of the branch currents of the replaced pulse voltage sources.

In this method, we perform the transient analysis to a linear resistive circuit only once. Hence, it is much simpler than the method using DC analysis. If we already have the netlist of the original circuit N, then we can obtain the netlist of $\hat{N}_p$ through a small modification to that of N.

In the above discussion, we have assumed that all nonlinear resistors of N are voltage-controlled. In the case that nonlinear resistors are current-controlled, we replace each nonlinear resistor by a pulse current source and perform the similar procedure. Then, the vector $\boldsymbol{s}$ and the matrix $\boldsymbol{H}$ are obtained from the branch voltages of the pulse sources.

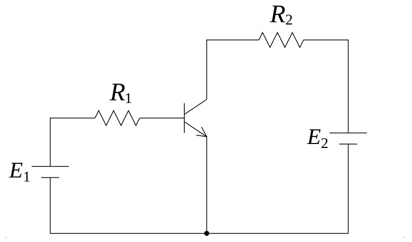

Fig. 3. Example circuit 1.

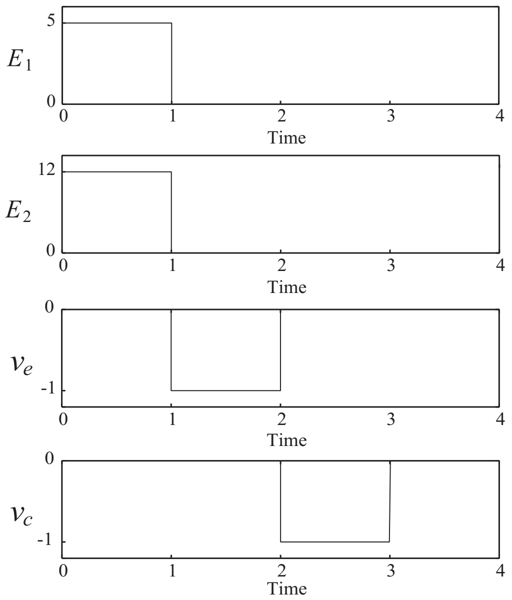

Fig. 4. Branch voltages of the pulse voltage sources (Example 1).

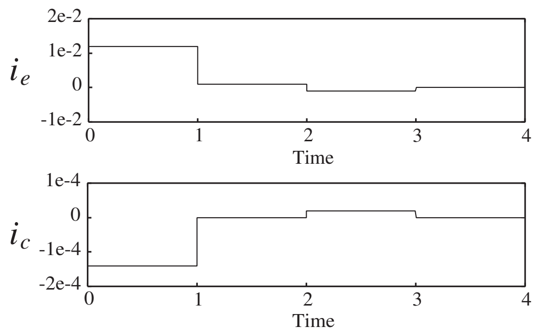

Fig. 5. Branch currents of the pulse voltage sources (Example 1).

Example 1: Consider the simple transistor circuit shown in Fig. 3 [3] where $E_1 = 5$, $E_2 = 12$, $R_1 = 100$K, and $R_2 = 1$K. We use the Ebers-Moll model

$$\begin{bmatrix} i_e \\ i_c \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ -0.99 & 1 \end{bmatrix} \begin{bmatrix} g_1(v_e) \\ g_2(v_c) \end{bmatrix} \tag{6}$$

and consider the diodes in the Ebers-Moll model as nonlinear resistors. Replace the independent voltage sources and the nonlinear resistors by pulse voltage sources with the branch voltages as shown in Fig. 4 and obtain the linear circuit $\hat{N}_p$. Performing the transient analysis of SPICE to $\hat{N}_p$, we obtain the branch currents of the pulse voltage sources as shown in Fig. 5. From the values on the time interval $[0, 1]$, $\boldsymbol{s}$ is obtained, and from the values on the time intervals $[1, 2]$ and $[2, 3]$, $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are obtained, respectively. Hence, the hybrid equations are given as follows:

$$\begin{bmatrix} g_1(v_e) \\ g_2(v_c) \end{bmatrix} + \begin{bmatrix} 1.009 \times 10^{-3} & -9.900 \times 10^{-4} \\ -1.980 \times 10^{-7} & 1.980 \times 10^{-5} \end{bmatrix} \begin{bmatrix} v_e \\ v_c \end{bmatrix} - \begin{bmatrix} 1.198 \times 10^{-2} \\ -1.396 \times 10^{-4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7}$$

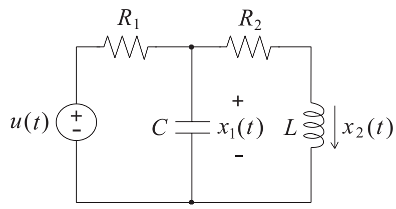

Fig. 6. Example circuit 2.

IV. Formulating State Equations Using SPICE

In this section, we show that state equations of nonlinear dynamic circuits containing capacitors and/or inductors can also be formulated by the proposed method.

We first consider a linear RLC circuit with l inputs and m capacitors and/or inductors. Then, the state equations can be written as follows [1]:

$$\boldsymbol{C}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \tag{8}$$

where $\boldsymbol{x}(t)$ is an m-dimensional variable vector consisting of the branch voltages of capacitors and/or the branch currents of inductors, $\boldsymbol{A}$ is an $m \times m$ matrix, $\boldsymbol{B}$ is an $m \times l$ matrix, $\boldsymbol{C}$ is an $m \times m$ diagonal matrix whose diagonal elements take the values of the capacitances and/or inductances, $\boldsymbol{u}(t)$ is an l-dimensional vector accounting for the independent and/or time-dependent sources, and t denotes the time.

In the proposed method, we first replace the capacitors and inductors by pulse voltage sources and pulse current sources, respectively, where the value of the jth pulse source is 1 on the time interval $[j-1, j]$ and 0 on other time intervals ($j = 1, 2, \cdots, m$). Then, we replace the jth independent or time-dependent source by a pulse source with the value 1 on the time interval $[m+j-1, m+j]$ and 0 on other time intervals ($j = 1, 2, \cdots, l$), and consider the linear *resistive* circuit $\hat{N}_p$ thus obtained. By performing the transient analysis to $\hat{N}_p$, we can obtain the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

Example 2: Consider the linear RLC circuit shown in Fig. 6, where $C = 1$, $L = 1$, $R_1 = 2$, and $R_2 = 3$. Performing the transient analysis of SPICE to the linear resistive circuit $\hat{N}_p$, we obtain the branch currents/voltages of the pulse voltage/current sources, respectively, as shown in Figs. 7 and 8. Hence, the state equations are given as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.5 & -1.0 \\ 1.0 & -3.0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(t). \tag{9}$$

We next consider a nonlinear dynamic circuit with l inputs, m capacitors and/or inductors, and n nonlinear resistors. Replace the voltage-controlled nonlinear resistors and capacitors

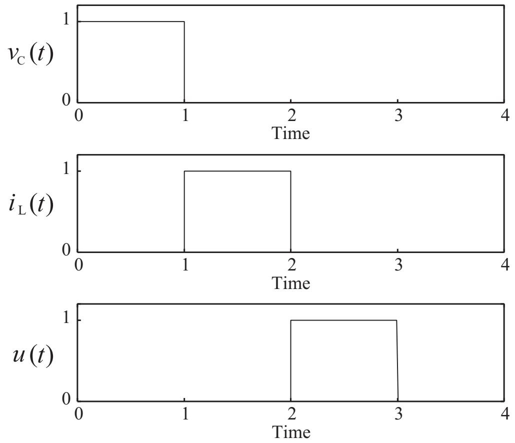

Fig. 7. Branch voltages or currents of the pulse sources (Example 2).

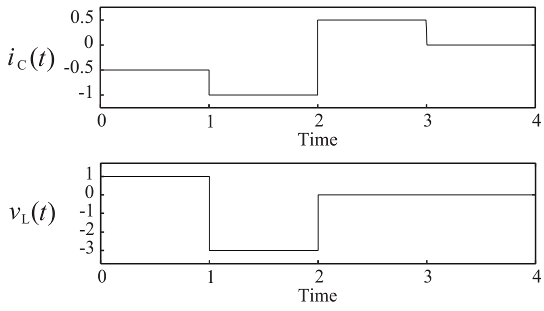

Fig. 8. Branch current or voltage of the pulse sources (Example 2).

by pulse voltage sources, the current-controlled nonlinear resistors and inductors by pulse current sources, and the independent and/or time-dependent sources by pulse sources in a similar way and obtain a linear circuit $\hat{N}_p$. By performing the transient analysis to $\hat{N}_p$, we obtain the state equations of the form:

$$\boldsymbol{C}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_{11}\boldsymbol{x}(t) + \boldsymbol{A}_{12}\boldsymbol{y}(t) + \boldsymbol{B}_1\boldsymbol{u}(t) \quad (10)$$

$$\boldsymbol{g}[\boldsymbol{y}(t)] = \boldsymbol{A}_{21}\boldsymbol{x}(t) + \boldsymbol{A}_{22}\boldsymbol{y}(t) + \boldsymbol{B}_2\boldsymbol{u}(t), \quad (11)$$

where $\boldsymbol{x}(t)$ is an m-dimensional variable vector consisting of branch voltages of capacitors and/or branch currents of inductors, $\boldsymbol{y}(t)$ is an n-dimensional variable vector consisting of branch voltages and/or currents of nonlinear resistors, $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{12}$, $\boldsymbol{A}_{21}$, and $\boldsymbol{A}_{22}$ are $m \times m$, $m \times n$, $n \times m$, and $n \times n$ matrices, respectively, and $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are $m \times l$ and $n \times l$ matrices, respectively.

From (11), an implicit function $\boldsymbol{y}(t) = \boldsymbol{G}(\boldsymbol{x}(t), t)$ is defined. Hence, (10) can be considered as a state equation of the form $\boldsymbol{C}\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t)$ that contains an implicit function. Thus, using the idea of the multilevel Newton algorithm in [14], we can apply various implicit numerical integration methods (such as the implicit Runge-Kutta method) to the nonlinear state equations.

Example 3: We applied the proposed method to the DC power supply circuit shown in Fig. 9 [15]. By the topological restriction of SPICE, we considered that the diode is current-

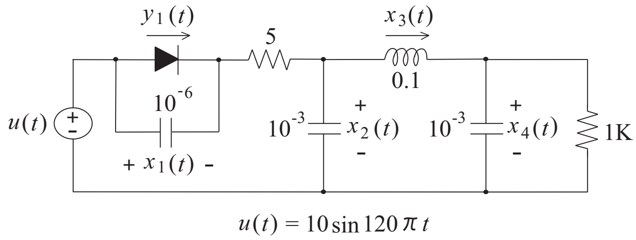

Fig. 9. Example circuit 3.

controlled. Applying the proposed method, we obtain the state equations as follows:

$$\begin{bmatrix} 10^{-6}\dot{x}_1 \\ 10^{-3}\dot{x}_2 \\ 10^{-1}\dot{x}_3 \\ 10^{-3}\dot{x}_4 \\ g_1(y_1) \end{bmatrix} = \begin{bmatrix} -0.2 & -0.2 & 0 & 0 & -1 \\ -0.2 & -0.2 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -0.001 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.2 \\ 0 \\ 0 \\ 0 \end{bmatrix} 10 \sin 120\pi t. \quad (12)$$

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