2008 International Symposium on Nonlinear Theory and its Applications
NOLTA'08, Budapest, Hungary, September 7-10, 2008

Numerical Bifurcation Analysis of Systems with Variable Switching Conditions

Quentin Brandon|, Tetsushi Ueta|, Takuji Kousaka† and Danièle Fournier-Prunaret‡

| Center for Advanced Information Technology
Faculty of Engineering, Tokushima University
2-1 Minami-Josanjima, Tokushima, 770-8506 JAPAN
†Dept. of Mechanical and Energy Systems Engineering
Faculty of Engineering, Oita University
700 Dannoharu, Oita, 870-1192 JAPAN
‡LATTIS-INSA, Université de Toulouse
135 Avenue de Rangueil, Toulouse, 31077 France
Email: {qbrandon,tetsushi}@is.tokushima-u.ac.jp , takuji@cc.oita-u.ac.jp, daniele.fournier@insa-toulouse.fr

Abstract—In this paper, we consider a method of numerical bifurcation analysis of nonlinear switched autonomous systems. Its development is motivated by several factors: linear systems can be studied through analytical approaches; smooth and non-autonomous systems can be studied using a standard periodical Poincaré map. Our goal is to develop a method as general as possible and implement it as a computer-based tool. The main evolution we will describe this time is the possible treatment of variable switching conditions.

1. Introduction

Switched systems exist in many domains such as mechanics or electronics. The solution function is usually continuous but presents points of non-derivability where discrete changes occur. Some of the most widely studied applications are related to electrical engineering, such as power converters investigated as piecewise smooth systems by di Bernardo [1], Tse [2] or Banerjee [3][4]; or some PLL models as introduced by Acco [5], referring to this type of model as "hybrid sequential." Little work has been done in the analysis of nonlinear piecewise smooth models, apart from Kawakami, Ueta and Kousaka [7][8]. As for hybrid linear models, they can be analyzed using rigorous analytical methods as shown by Kabe [6]. However the nonlinear property of many systems reserve this option for approximated models only. Proper treatment of nonlinearity requires numerical methods.

We review the analysis method based on a Poincaré map and introduce some modifications and improvements. Kousaka [7] introduced the steps to compute bifurcation sets of nonsmooth dynamical systems by solving a pure initial value problem (using an ODE solver only). On the other hand, in order to bring the equation sets of the model to an acceptable level of complexity, many assumptions were made, particularly on the Poincaré sections (perpendicular or parallel to the local coordinates), as in the analysis of the 3-state Alpazur oscillator [9] for instance. In order to generalize our method, we introduced two main evolutions. The first one is a numerical differentiation which lifts the necessity of expressing complex second derivatives. The second one is a generalization of the expression of the Poincaré section, which can be now represented as any smooth surface in the local referential. Such simplifications naturally ease the analysis process of systems with many discrete states. We consider the analysis results of the Alpazur oscillator with variable switching conditions. Our method here proves capable of handling switching conditions defined by non-trivial equations of multiple state variables.

2. Principles

2.1. Modeling the System

Let us consider a system written by a set of differential equations defined by smooth functions piecewisely, i.e., for the state i:

$$\frac{dX}{dt} = f_i(X), \qquad i = 1, \cdots m \tag{1}$$

where,

$$X(t) = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix} \in R^n. \tag{2}$$

Within each state there is a solution function such as:

$$X(t) = \varphi_i(t, X_i) \quad \text{with} \quad X(0) = X_i, \tag{3}$$

where X_i is the initial value of state i.
Now, based on the switching rules and the definition of a period of our system, we set the Poincaré map, placing its sections at the switching points. We assume for now that the switching conditions for each state can be expressed as a function of the system variables, e.g., for the state i: $q_i(X) = 0$.
The map is therefore expressed as:

$$\begin{aligned} \Pi_i &= \{X_i \in R^n \mid q_i = 0\} \\ T_i :\ \ & \Pi_i \to \Pi_{i+1} \\ & X_i \mapsto X_{i+1} = \varphi_i(\tau_i, X_i). \end{aligned} \tag{4}$$

We are now able to perform a local analysis over each partial orbit delimited by those Poincaré sections. Thus the Poincaré mapping is defined as a differentiable map:

$$T = T_m \circ \cdots \circ T_1 \circ T_0. \quad (5)$$

We finally apply a projection from R^n to R^{n-1} due to the equation of the final state switch condition $q_{m-1} = 0$ as seen in [7]:

$$p \; : \; \begin{array}{l} \Pi_0 \to \Sigma_0 \\ X_0 \mapsto U_0 \end{array} \quad \text{with} \quad T_l = p^{-1} \circ T \circ p, \quad (6)$$

which we use as a discrete definition of our system Fig.1.

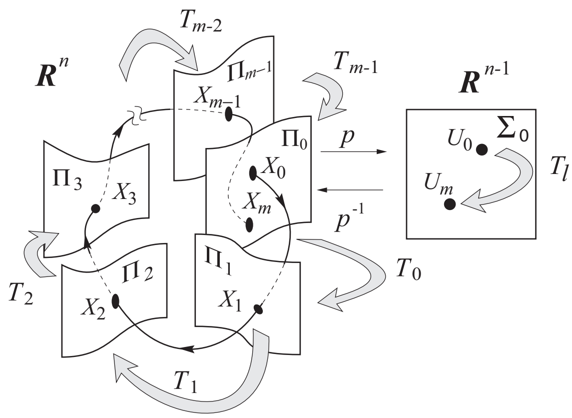

Figure 1: Abstract representation of the Poincaré map

2.2. Analysis approach

In order to compute phase portraits for a given system, we simply integrate the differential equations. The switching conditions previously defined are constantly verified in order to use the variational equations matching the current state. A proper adaptive stepsize Runge Kutta integration gave us good performance along with precious control of the numerical error.
For the computation of fixed points, we integrate the partial derivatives of the differential equations in parallel of the solution:

$$\frac{d}{dt}\frac{\partial \varphi_i}{\partial X_{i-1}} = \frac{\partial f_i}{\partial X}\frac{\partial \varphi_i}{\partial X_{i-1}} \quad \text{where} \quad \frac{\partial \varphi_i}{\partial X_{i-1}}(0) = I, \quad (7)$$

where I is the identity matrix. We compute the Jacobian matrix of each local map:

$$\frac{\partial X_i}{\partial X_{i-1}} = \frac{\partial \varphi_i}{\partial X_{i-1}} + \frac{dX_i}{dt}\frac{\partial \tau_i}{\partial X_{i-1}}, \quad (8)$$

The expression of $\frac{\partial \tau_i}{\partial X_{i-1}}$ depends on the switching condition, so its expression takes into account the following terms and functions: $\frac{\partial \varphi_i}{\partial X_{i-1}}$, $q_i(X)$, and $f_i(X)$. Next, we can express the Poincaré map:

$$\frac{\partial X_m}{\partial X_0} = \prod_{i=1}^{m} \frac{\partial X_i}{\partial X_{i-1}}. \quad (9)$$

We then apply the projection p and obtain the Jacobian:

$$DT_l(U_0) = \frac{\partial U_m}{\partial U_0}. \quad (10)$$

For a fixed point we solve:

$$T_l(U_0) = U_m = U_0. \quad (11)$$

We use the approximation of the tangent to the solution DT_l in the Newton method in order to compute an accurate value of U_0. If our initial value is in the domain of convergence, we can compute an accurate solution of the fixed point.
Finally, concerning the critical values required to compute bifurcation sets, we use the characteristic equation of our system in order to determine an extra constraint. Depending on the desired bifurcation diagram, we choose λ, one of the parameters from the diagram space, as an extra degree of freedom to compute a solution of our new equations set:

$$\chi_l(\mu) = \det(DT_l - \mu I_{n-1}) = 0, \quad (12)$$

where I_{n-1} is thye identity matrix. μ is defined by the bifurcation type. So our problem can be written:

$$F(U_0, \lambda) = \begin{bmatrix} U_m - U_0 \\ \chi_l(\mu) \end{bmatrix} = 0. \quad (13)$$

Which means we are to compute the following Jacobian matrix of Eq. (13):

$$\begin{bmatrix} \frac{\partial U_m}{\partial U_0} - I_{n-1} & \frac{\partial U_m}{\partial \lambda} \\ \frac{\partial DT_l}{\partial U_0} & \frac{\partial DT_l}{\partial \lambda} \end{bmatrix} \quad (14)$$

The second row of Eq. (14) requires the second derivatives of the solutions of variational Eq. (7). Computing a number of second derivatives can be done two ways: one based on an analytical approach, consisting in simply deriving once more the first derivative elements; the second one is numerical since it consists in approaching the tangent by differentiation, performing a multiple integration using shifted input variables.

3. Variably Switching Alpazur Oscillator

3.1. Model description

We use the Alpazur oscillator [8] with modified switching conditions.
The continuous variable is represented by: $X = \begin{pmatrix} x \\ y \end{pmatrix}$
According to the position of the switch, we have, for each state ($i = \{1, 2\}$), the following differential equations:
$f_i(X) = \begin{pmatrix} f_i(x, y) \\ g_i(x, y) \end{pmatrix}$

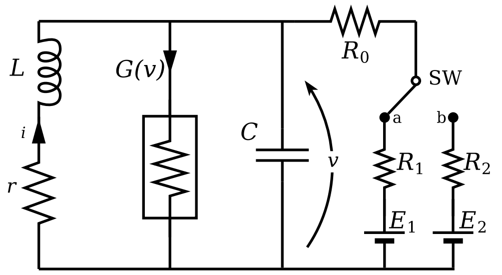

Figure 2: Electronic implementation of the Alpazur osc.

For state i: (terminal a($i=1$) or b($i=2$) in SW)

$$\begin{cases} \dfrac{dx}{dt} = f_i(x,y) = -rx - y \\ \dfrac{dy}{dt} = g_i(x,y) = x + (1-A_i)y - \dfrac{1}{3}y^3 + B_i. \end{cases} \tag{15}$$

where x and y are normalized variables corresponding to the current i and voltage v (detailed in [8]); and where A_i and B_i are real numbers.
In order to determine the discrete behavior, we establish the following switching rules:

$$\begin{aligned} q_1(x,y) &= y + 1.0 - 0.2\sin x \\ q_2(x,y) &= y + 0.1 - 0.2x^2, \end{aligned} \tag{16}$$

Such switching conditions are indeed very unlikely, but they demonstrate the efficiency of the method even for complex switching cases. This system exhibits chaotic orbits at particular parameter values such as the following set:
$r = 0.1 \quad A_1 = 0.2 \quad A_2 = 2.0 \quad B_1 = -0.2 \quad B_2 = 1.0$

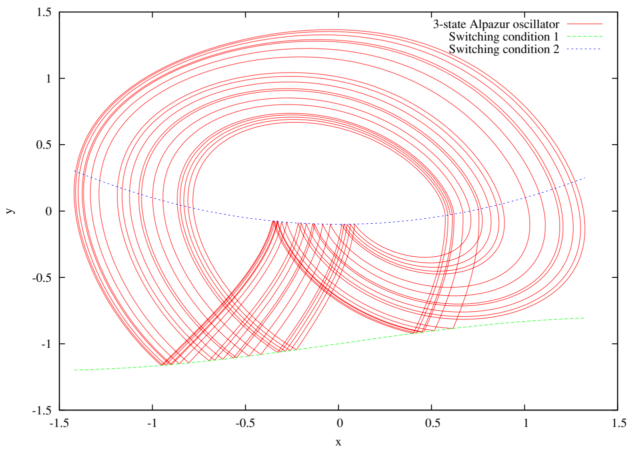

Figure 3: Sample phase portrait of chaotic behavior

We define the map:

$$\begin{aligned} T = T_1 \circ T_0 \quad : \quad \Pi_0 &\rightarrow \Pi_0 \\ x_0 &\mapsto x_2 = \varphi(\tau, x_0). \end{aligned} \tag{17}$$

3.2. Fixed points

As previously shown, the problem of Fixed points is as follows:

$$x_2 - x_0 = 0. \tag{18}$$

In order to achieve the appropriate correction, we need to compute:

$$\begin{aligned} DT_l &= \frac{\partial x_2}{\partial x_0} \\ &= \frac{\partial x_2}{\partial X_1}\frac{\partial X_1}{\partial X_0}\frac{\partial X_0}{\partial x_0} \\ &= \frac{\partial x_2}{\partial x_1}\frac{\partial x_1}{\partial X_0}\frac{\partial X_0}{\partial x_0} + \frac{\partial x_2}{\partial y_1}\frac{\partial y_1}{\partial X_0}\frac{\partial X_0}{\partial x_0} \\ &= \frac{\partial x_2}{\partial x_1}\left(\frac{\partial x_1}{\partial x_0} + \frac{\partial x_1}{\partial y_0}\frac{\partial y_0}{\partial x_0}\right) + \frac{\partial x_2}{\partial y_1}\left(\frac{\partial y_1}{\partial x_0} + \frac{\partial y_1}{\partial y_0}\frac{\partial y_0}{\partial x_0}\right). \end{aligned} \tag{19}$$

For each State i we compute:

$$\begin{cases} \dfrac{\partial x_i}{\partial x_{i-1}} = \dfrac{\partial \varphi_i}{\partial x_{i-1}} + f_i(x_i,y_i)\dfrac{\partial \tau_i}{\partial x_{i-1}} \\ \dfrac{\partial \tau_i}{\partial x_{i-1}} = -\dfrac{\dfrac{\partial \varphi_i}{\partial x_{i-1}}q_{iy} - \dfrac{\partial \phi_i}{\partial x_{i-1}}q_{ix}}{f_i(x_i,y_i)q_{iy} - g_i(x_i,y_i)q_{ix}} \end{cases} \tag{20}$$

$$\begin{cases} \dfrac{\partial x_i}{\partial y_{i-1}} = \dfrac{\partial \varphi_i}{\partial y_{i-1}} + f_i(x_i,y_i)\dfrac{\partial \tau_i}{\partial y_{i-1}} \\ \dfrac{\partial \tau_i}{\partial y_{i-1}} = -\dfrac{\dfrac{\partial \varphi_i}{\partial y_{i-1}}q_{iy} - \dfrac{\partial \phi_i}{\partial y_{i-1}}q_{ix}}{f_i(x_i,y_i)q_{iy} - g_i(x_i,y_i)q_{ix}} \end{cases}, \tag{21}$$

where $\partial y_i/\partial x_i = \partial y_i/\partial x|_{q_i(x_i,y_i)=0 \text{ and } x=x_i}$; q_{ix} and q_{iy} are the components of a vector tangent to the switching curve at X_i.
Then there are two ways of calculating $\partial y_i/\partial x_{i-1}$ and $\partial y_i/\partial y_{i-1}$:

$$\begin{aligned} \frac{\partial y_i}{\partial x_{i-1}} &= \frac{\partial \phi_i}{\partial x_{i-1}} + g_i(x_i,y_i)\frac{\partial \tau_i}{\partial x_{i-1}} \\ &= \frac{\partial y_i}{\partial x_i}\frac{\partial x_i}{\partial x_{i-1}} \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{\partial y_i}{\partial y_{i-1}} &= \frac{\partial \phi_i}{\partial y_{i-1}} + g_i(x_i,y_i)\frac{\partial \tau_i}{\partial y_{i-1}} \\ &= \frac{\partial y_i}{\partial x_i}\frac{\partial x_i}{\partial y_{i-1}} \end{aligned}. \tag{23}$$

We numerically integrate the required elements:

$$\begin{aligned} &\frac{d}{dt}\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} f_i(x,y) \\ g_i(x,y) \end{bmatrix} \quad \begin{matrix} \text{State 1: from } x_0 \\ \text{State 2: from } x_1 \end{matrix} \\ &\frac{d}{dt}\begin{bmatrix} \dfrac{\partial \varphi_i}{\partial x_{i-1}} & \dfrac{\partial \varphi_i}{\partial y_{i-1}} \\ \dfrac{\partial \phi_i}{\partial x_{i-1}} & \dfrac{\partial \phi_i}{\partial y_{i-1}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_i}{\partial x} & \dfrac{\partial f_i}{\partial y} \\ \dfrac{\partial g_i}{\partial x} & \dfrac{\partial g_i}{\partial y} \end{bmatrix}\begin{bmatrix} \dfrac{\partial \varphi_i}{\partial x_{i-1}} & \dfrac{\partial \varphi_i}{\partial y_{i-1}} \\ \dfrac{\partial \phi_i}{\partial x_{i-1}} & \dfrac{\partial \phi_i}{\partial y_{i-1}} \end{bmatrix} \\ &\text{where } \begin{bmatrix} \dfrac{\partial \varphi_i}{\partial x_{i-1}} & \dfrac{\partial \varphi_i}{\partial y_{i-1}} \\ \dfrac{\partial \phi_i}{\partial x_{i-1}} & \dfrac{\partial \phi_i}{\partial y_{i-1}} \end{bmatrix}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{24}$$

We now use the Newton method to compute the correction to be applied:

$$x_0' = x_0 - \frac{x_2 - x_0}{\dfrac{\partial x_2}{\partial x_0} - 1}. \tag{25}$$

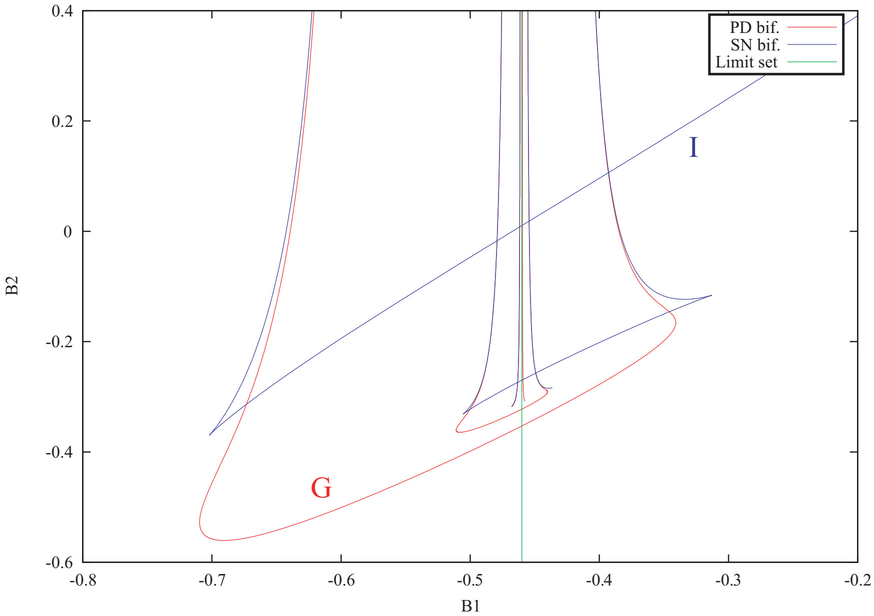

Figure 4: Bifurcation diagram in the B_1/B_2 parameter plan

3.3. Bifurcations

The characteristic equation is fairly simple:

$$\chi(\mu) = \det(DT_l - \mu) = 0, \tag{26}$$

hence the Jacobian matrix:

$$\begin{bmatrix} \dfrac{\partial x_2}{\partial x_0} - 1 & \dfrac{\partial x_2}{\partial \lambda} \\ \dfrac{\partial DT_l}{\partial x_0} & \dfrac{\partial DT_l}{\partial \lambda} \end{bmatrix} \tag{27}$$

We obtained the bifurcation diagram Fig.4. Note that these results are consistant with the results from the other versions of the Alpazur oscillator: 2-state and 3-state. We find the same monotonic bifurcation structure with a limit set at a specific value of B_1.

4. Generalization

This method can naturally be applied to systems with higher dimension. This corresponds to applying a projection on a multi-dimensional surface instead of a simple curve. We approximate the surface locally using a plane (in 2D we approximate the switching curve by a line), and compute the intersection of the variational vector with this plane: The plane is defined by it normal: $\vec{n} \mid X_i.\vec{n} = k$ then we can compute $\partial\tau_i/\partial X_{i-1}$

$$\frac{\partial \tau_i}{\partial X_{i-1}} = \frac{k - \left(X_i + \dfrac{\partial X_i}{\partial X_{i-1}}\right).\vec{n}}{F(X_i).\vec{n}} \tag{28}$$

5. Conclusion

We have detailed a procedure of numerical bifurcation analysis of switched systems with great flexibility towards the nature of the switching conditions, and illustrated it with the results of a modified Alpazur oscillator. Our future work will include studying other systems with higher dimension, and explore the possible switching scenarios, which must be expressed in a generic form in order to design a flexible computer tool based on this method.

References

[1] di Bernardo, M.; Tse, C., *Chaos in Circuits and Systems*, Series B, Vol. 11, pp. 317–340, 2002.

[2] Tse, C., *Complex Behavior of Switching Power Converters*, pp. 57–80, 2004.

[3] Banerjee, S.; Chakrabarty, K., *IEEE Trans. Power Electronics*, Vol. 13, No. 2, pp. 252–260, Mar. 1998.

[4] Banerjee, S.; Karthik, M.S.; Yuan, G.H.; Yorke, J.A., *IEEE Trans. Circuits and Systems I*, Vol. 47, No. 3, pp. 389–394, Mar. 2000.

[5] Acco, P., *PhD. thesis INSAT*, No. 707, 2003.

[6] Kabe, T.; Parui, S; Torikai, H.; Banerjee, S; Saito, T., *IEICE Trans. Fundamentals of Electronics, Communications and Comp. Sci.*, E90-A, 2, pp. 448–456, 2007.

[7] Kousaka, T.; Ueta, T.; Kawakami, H., *IEEE Trans. Circuits and Systems II: Analog and Digital Signal Processing*, Vol. 46, No. 7, pp. 878–885, Jul. 1999.

[8] H. Kawakami and R. Lozi, *Proc. RIMS Conf. "Structure and Bifurcations of Dynamical Systems," ed. by S. Ushiki, World Scientific*, pp.39–58, 1992

[9] Brandon, Q.; Ueta, T.; Kousaka, T.; Fournier-Prunaret, D., *NOLTA 2007*, pp. 160–163, 2007