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Bifurcation analysis of Izhikevich model

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Abstract—A simple model of spiking neurons is proposed by Izhikevich. Some experimental results, origin of the mathematical model, firing pattern of all known types of neurons are reported. However bifurcation analysis has not been investigated in details. In this paper, we propose a method to trace bifurcation sets for a piecewise nonlinear equations with state-dependent jump and investigate bifurcations of equilibrium and periodic solutions for this system.

1. Introduction

There are two main approaches in biological fields recently. The first method consists in an actual experimental analysis. The second method consists in development of mathematical models based on available data and knowledge of dynamical systems. As the experimental technique develops, a great number of genetic data and the data of the model organism have been accumulated. According to this, many mathematical models have been proposed. In order to verify a model, the results of the simulation should match the experimental results. For the progression of biological science, the focus is put on the development of mathematical models.

A simple model of spiking neurons is proposed by Izhikevich[1]. The biologically plausibility of the model is as good as that of the Hodgkin-Huxley-type model. Some experimental results, by changing parameters, firing pattern of all known types of neurons are illustrated. However bifurcation analysis has not been investigated in detail. In this paper, first of all, we propose a method to trace bifurcation sets for a piecewise nonlinear equations with state-dependent jump. We define a piecewise-defined system, its solution and limit cycle. Next, we investigate bifurcations of equilibrium and periodic solutions for this system. Bifurcation diagrams are obtained numerically and chaotic regions are clarified.

2. Analyzing method

Let us consider m autonomous differential equations

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f_k}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\lambda_k}), \quad k = 0, 1, 2, \cdots, m-1 \tag{1}$$

where $t \in \boldsymbol{R}$, $\boldsymbol{x} \in \boldsymbol{R}^n$. $\boldsymbol{\lambda} \in \boldsymbol{R}^r$ is an invariant parameter for $\boldsymbol{f}_0$, $\boldsymbol{f}_1$, $\cdots \boldsymbol{f}_{m-1}$ and $\boldsymbol{\lambda} \in \boldsymbol{R}^s$ is a parameter depending only on $\boldsymbol{f}_k$. r and s are integers. We call these equations piecewise-defined differential equations. Assume that $\boldsymbol{f}_k$ is C^∞-class map for all variables and parameters and every equation in Eq.(1) has a solution with an arbitrary initial value $\boldsymbol{x}_{k0}$, such that

$$\boldsymbol{x}_k(t) = \boldsymbol{\varphi}_k(t, \boldsymbol{x}_{k0}), \quad \boldsymbol{x}_k(0) = \boldsymbol{x}_{k0} \tag{2}$$

Assume also that the function changes from $\boldsymbol{f}_k$ to $\boldsymbol{f}_{k+1}$ when a solution $\boldsymbol{\varphi}_k$ starting from Π_k reaches Π_{k+1} with the time τ_k. Thus

$$\boldsymbol{x}_{k+1}(t) = \boldsymbol{\varphi}_k(t, \boldsymbol{x}_k) \tag{3}$$

where

$$\boldsymbol{x}_{k+1}(0) = \boldsymbol{\varphi}_{k+1}(0, \boldsymbol{x}_{k+1}) = \boldsymbol{\varphi}_k(\tau_k, \boldsymbol{x}_k) \tag{4}$$

Then a periodic solution (limit cycle) is written as follows:

$$\boldsymbol{x}_0 = \boldsymbol{x}_m = \boldsymbol{\varphi}_{m-1}(\tau_{m-1}, \boldsymbol{x}_{m-1}) \tag{5}$$

Note that the solution Eq.(5) is continuous, but not differentiable for all states. We place local section for this limit cycle at every break point defined by the following scalar function q_k:

$$\Pi_k = \{\boldsymbol{x}_k \in \boldsymbol{R}^n \mid q_k(\boldsymbol{x}_k) = 0\}, \; k = 0, 1, 2, \cdots, m-1 \tag{6}$$

The following local mappings are defined:

$$\begin{aligned}
T_0 &: \Pi_0 \to \Pi_1 \\
&\quad \boldsymbol{x}_0 \mapsto \boldsymbol{x}_1 = \boldsymbol{\varphi}_0(\tau_0, \boldsymbol{x}_0) \\
T_1 &: \Pi_1 \to \Pi_2 \\
&\quad \boldsymbol{x}_1 \mapsto \boldsymbol{x}_2 = \boldsymbol{\varphi}_1(\tau_1, \boldsymbol{x}_1) \\
&\quad \cdots \\
T_{m-1} &: \Pi_{m-1} \to \Pi_0 \\
&\quad \boldsymbol{x}_{m-1} \mapsto \boldsymbol{x}_0 = \boldsymbol{\varphi}_{m-1}(\tau_{m-1}, \boldsymbol{x}_{m-1})
\end{aligned} \tag{7}$$

Poincaré mapping is defined as a differentiable composite map descried by

$$T = T_0 \circ T_1 \circ \cdots T_{m-1} \tag{8}$$

Hence, the period of the limit cycle τ is obtained by

$$\tau = \sum_{k=0}^{m-1} \tau_k \tag{9}$$

The derivative with the initial value of the Poincaré map is given by

$$\left.\frac{\partial T}{\partial \boldsymbol{x}_0}\right|_{t=\tau} = \Pi_{k=0}^{m-1} \left.\frac{\partial T_k}{\partial \boldsymbol{x}_k}\right|_{t=\tau_k} \tag{10}$$

Each Jacobian matrix can be written as follows:

$$\frac{\partial T_k}{\partial \boldsymbol{x}_k} = \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} + \frac{\partial \boldsymbol{\varphi}_k}{\partial t}\frac{\partial \tau_k}{\partial \boldsymbol{x}_k} = \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} + \boldsymbol{f}_k \frac{\partial \tau_k}{\partial \boldsymbol{x}_k} \tag{11}$$

We should remark that the function

$$q_k(\boldsymbol{x}_k) = q_k(\boldsymbol{\varphi}_k(\tau_k, \boldsymbol{x}_k)) = 0 \tag{12}$$

is differentiable for $\boldsymbol{x}_k$. Thereby

$$\frac{\partial q_k}{\partial \boldsymbol{x}_k}\left(\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} + \boldsymbol{f}_k \frac{\partial \tau_k}{\partial \boldsymbol{x}_k}\right) = 0 \tag{13}$$

then we have the following relationship from Eq.(13)

$$\frac{\partial \tau_k}{\partial \boldsymbol{x}_k} = -\frac{1}{\dfrac{\partial q_k}{\partial \boldsymbol{x}}\boldsymbol{f}_k}\frac{\partial q_k}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} \tag{14}$$

By substituting Eq.(14) into Eq.(11), we have

$$\begin{aligned}\frac{\partial T_k}{\partial \boldsymbol{x}_k} &= \frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} - \frac{1}{\dfrac{\partial q_k}{\partial \boldsymbol{x}}\boldsymbol{f}_k}\boldsymbol{f}_k\frac{\partial q_k}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k} \\ &= \left[\boldsymbol{I}_n - \frac{1}{\dfrac{\partial q_k}{\partial \boldsymbol{x}}\boldsymbol{f}_k}\boldsymbol{f}_k\frac{\partial q_k}{\partial \boldsymbol{x}}\right]\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\end{aligned} \tag{15}$$

where I_n is an $n \times n$ identity matrix. $\partial \boldsymbol{\varphi}_k / \partial \boldsymbol{x}_k$ can be obtained by solving the following differential equation

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\right) = \frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}}\left(\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\right) \\ &\left.\frac{\partial \boldsymbol{\varphi}_k}{\partial \boldsymbol{x}_k}\right|_{t=0} = \boldsymbol{I}_n\end{aligned} \tag{16}$$

Now we define a local coordinate $\boldsymbol{w} \in \Sigma_0 \subset \boldsymbol{R}^{n-1}$ corresponding to Π_0 by using a projection p and embedding map p^{-1}

$$p^{-1} : \Sigma_0 \rightarrow \Pi_0, \quad p : \Pi_0 \rightarrow \Sigma_0 \tag{17}$$

Accordingly, the Poincaré mapping on the local coordinate is obtained as

$$\begin{aligned}T_\ell : \Sigma_0 &\rightarrow \Sigma_0 \\ \boldsymbol{w} &\mapsto p \circ T \circ p^{-1}(\boldsymbol{w})\end{aligned} \tag{18}$$

A fixed point of the Poincaré mapping is obtained by solving the following equation

$$T_\ell(\boldsymbol{w}) - \boldsymbol{w} = \boldsymbol{0} \tag{19}$$

The Jacobian matrix required Newton's method is give by

$$\frac{\partial T_\ell}{\partial \boldsymbol{w}_0} = DT_\ell(\boldsymbol{w}_0) = \frac{\partial p}{\partial \boldsymbol{x}}\frac{\partial T}{\partial \boldsymbol{x}_0}\frac{\partial p^{-1}}{\partial \boldsymbol{w}} \tag{20}$$

The characteristic equation for the fixed point is given by

$$\chi_\ell(\mu) = \det[DT_\ell - \mu I_{n-1}] = 0 \tag{21}$$

The roots of Eq.(21) give multipliers of the fixed points. We can obtain accurate location of the fixed point $\boldsymbol{w}$ and bifurcation parameter value λ by solving the following equation by Newton's method

$$F(\boldsymbol{w}, \lambda) = \begin{bmatrix} T_\ell(\boldsymbol{w}) - \boldsymbol{w} \\ \chi_\ell(\mu) \end{bmatrix} = 0 \tag{22}$$

3. An application for Izhikevich model

Izhikevich proposed a simple model of spiking neurons. There are two features in this model. First, it doesn't cost the calculation cost more than Hodgkin-Huxley-type model. Second, it confirmed a lot of firing patterns. The equation set are as follows:

$$\begin{cases} \dot{v} = 0.04v^2 + 5v + 140 - u + I \\ \dot{u} = a(bv - u) \end{cases} \tag{23}$$

then reset after it spikes:

$$\text{if } v \geq 30\,\text{mV}, \text{ then} \begin{cases} v \leftarrow c \\ u \leftarrow u + d \end{cases} \tag{24}$$

where, the state variables v and u correspond to the membrane potential of the neuron and membrane recovery variable, respectively. The parameters are a, b, c, d and I. Here, a, b, and I related with the time scale of the recovery variable, the sensitivity of the recovery variable u to v and Synaptic current, respectively. c and d are reset value. After the spike reaches its apex, the state variable v and u are reset according to Eq. (24).

In the following, we restate the method of analysis for this model. Let solutions be the following equations:

$$\begin{cases} v(t) = \varphi_0(t, \boldsymbol{u}_0, \lambda_0, \lambda) \\ u(t) = \phi_0(t, \boldsymbol{u}_0, \lambda_0, \lambda) \end{cases} \tag{25}$$

We place local section for the limit cycle at every break point defined:

$$\Pi_0 = \{(v, u) \in \boldsymbol{R}^2 \mid q(v, u) = v - V = 0\} \tag{26}$$

Local section and local mappings are defined as follows:

$$\begin{aligned}T_0 : \Pi_0 &\rightarrow \Pi_0 \\ v_0 &\mapsto v_1 = c \\ u_0 &\mapsto u_1 = \phi_0(t, \boldsymbol{u}_0, \lambda_0, \lambda) + d\end{aligned} \tag{27}$$

Thus, we have

$$T = T_0 \tag{28}$$

We choose the projection and embedding as follows:

$$\begin{aligned}&h : \Pi_0 \rightarrow \Sigma \quad \boldsymbol{x} = \begin{pmatrix} v \\ u \end{pmatrix} \mapsto \boldsymbol{u} = u \\ &h^{-1} : \Sigma \rightarrow \Pi_0 \quad \boldsymbol{u} = u \mapsto \boldsymbol{x} = \begin{pmatrix} V \\ u \end{pmatrix}\end{aligned} \tag{29}$$

The Jacobian matrix of the Poincaré mapping is as follows:

$$\begin{aligned} \frac{\partial T_\ell}{\partial \boldsymbol{u}_0} &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\dfrac{f_2}{f_1} & 1 \end{pmatrix} \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= -\frac{f_2}{f_1}\frac{\partial \varphi_1}{\partial u_0} + \frac{\partial \varphi_2}{\partial u_0} \end{aligned} \tag{30}$$

We can obtain the location of the fix point and bifurcation parameter value by applying Newton's method.

4. Bifurcation phenomena for Izhikevich model

We observed a rich variety of firing patterns, changing the parameters a, b, c and d by Eq. (23) and (24). For example, when $a = 0.02$, $b = 0.2$, $c = -65$, $d = 8$ and $a = 0.1$, $b = 0.2$, $c = -65$, $d = 2$ are set, it is regular spiking (RS) and fast spiking(FS). It is thought that the bifurcation to divide those firing patterns exists shown in Fig. 1. Then, we clarify that by showing the bifurcation of fixed point.

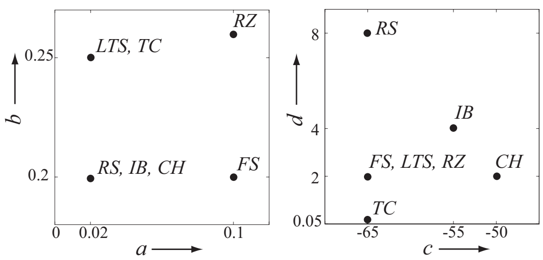

Figure 1: The firing patterns of neurons [1].

5. Bifurcation of fixed point

We compute bifurcation diagrams in the a-b plane by using the value of parameter c and d shown in Fig. 1. Figures 2 and 4 show the bifurcation diagrams with $I = 10$. Figure 2, 4 are the bifurcation diagrams fixed to $c = -55$, $d = 4$ and $c = -50$, $d = 2$ respectively. Hence, saddle-node bifurcation for an equilibrium is labeled by d and subscript of these symbols show numbers of unstable. Symbol I^k show period-doubling(PD) bifurcations of a k-periodic limit cycle.

In the Fig. 2 there exists an island surrounded by PD bifurcation curves. Inside this, the limit cycle(Fig. 3(a)) is bifurcated by PD cascade; I^i, $i = 1, 2, 4, \cdots \infty$. Figure 3(b) is two-periodic and Fig. 3(c) is four-periodic solution, respectively. Via PD cascade, we have chaotic attractor, see Fig. 3(d). For $b > 0.6$, bifurcation structure is not sensitive to variations of b.

In the Fig. 4, we see that there confirmed right-hand side region which has a limit cycle state. Since there exist PD cascades along the arrow(⇒), chaotic states are easily expected (Fig. 5). Moreover, period-adding can be confirmed by the arrow(→) and the attractor is shown in Fig. 5. Figure 6 is an expansion of Fig. 4.

The bifurcations of fixed point don't exist when the values of parameter c and d are fixed to $c = -65$, $d = 0.05$, $c = -65$, $d = 2$, and $c = -65 ! \$ d = 8$. When parameters $c = -65$, $d = 2$ are fixed, we observed fast spiking(FS) with $a = 0.1$, $b = 0.2$, low-threshold spiking(LTS) with $a = 0.02$, $b = 0.25$ and resonator(RZ) with $a = 0.1 ! \$ b = 0.26$. However, there are no bifurcation that divides those states. That is to say, the state of the transition is deeply related.

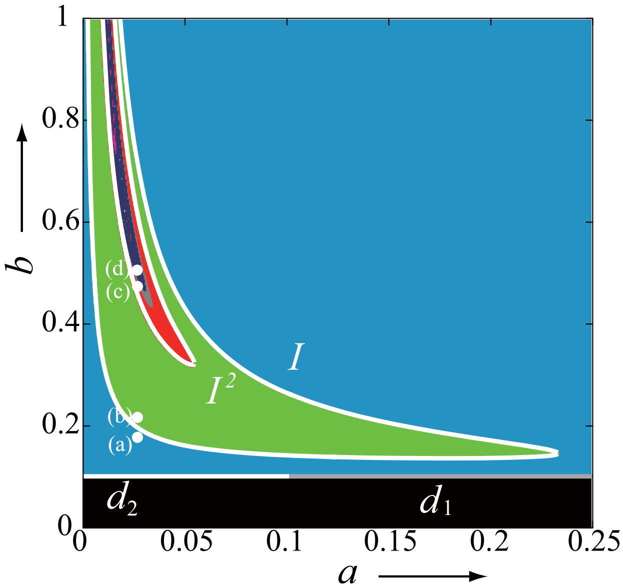

Figure 2: Bifurcation diagram int the a-b plane with $c = -55$, $d = 4$, $I = 10$.

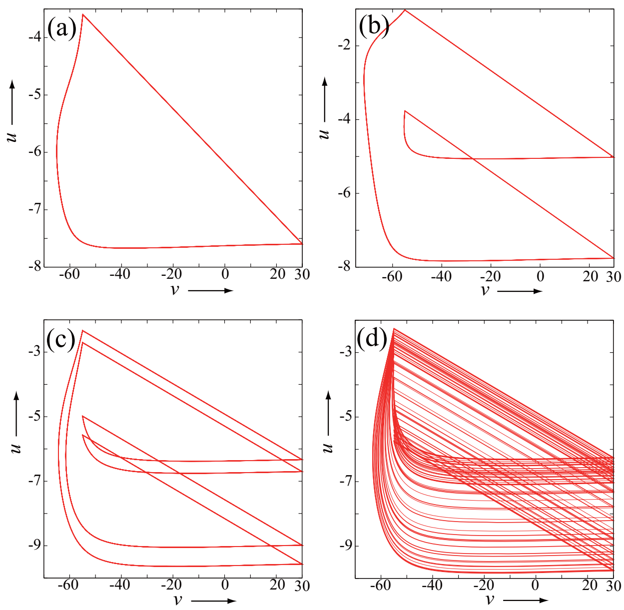

Figure 3: Phase portraits with $a = 0.025$, v-u plane. (a) Period-1 orbit, $b = 0.2$. (b) Period-2 orbit, $b = 0.21$. (c) Period-4 orbit, $b = 0.48$. (d) Chaos, $b = 0.55$.

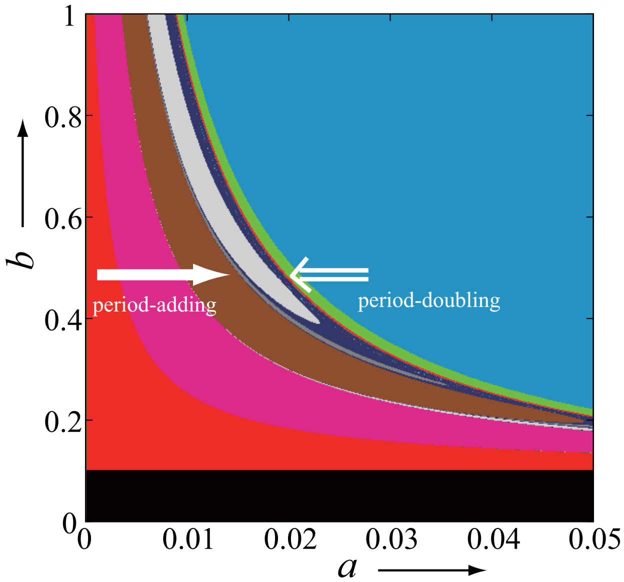

Figure 4: Bifurcation diagram int the a-b plane with $c = -50$, $d = 2$, $I = 10$.

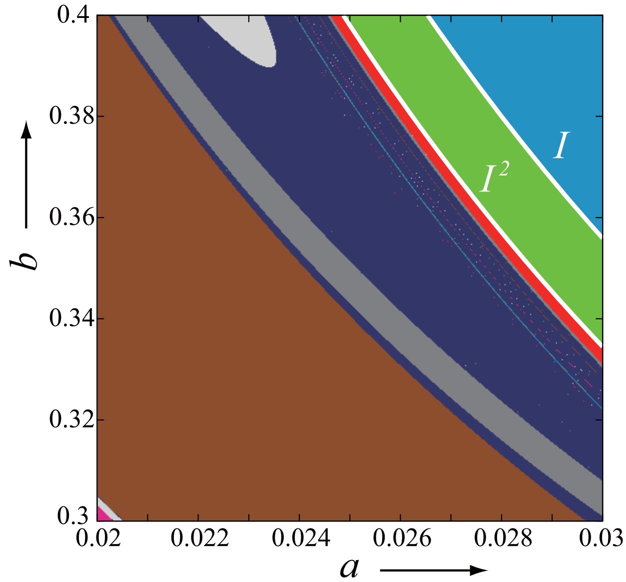

Figure 6: Bifurcation diagram int the a-b plane with $c = -50$, $d = 2$, $I = 10$.

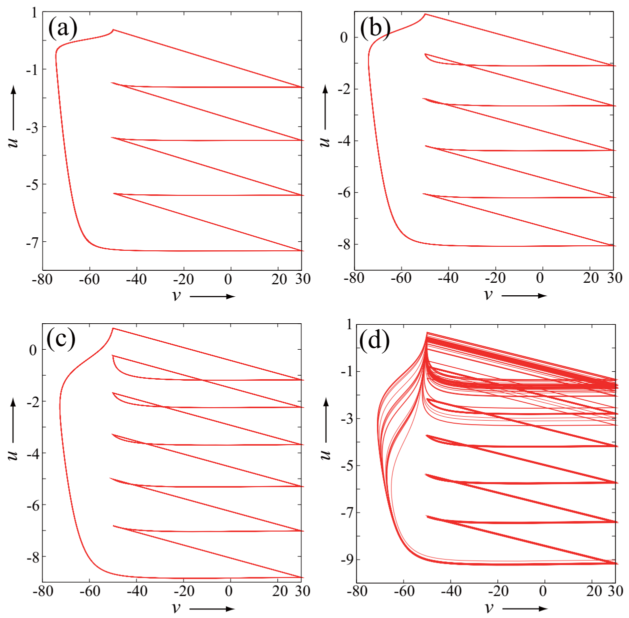

Figure 5: Phase portraits with $b = 0.5$, v-u plane. (a) Period-4 orbit, $a = 0.003$. (b) Period-5 orbit, $a = 0.007$. (c) Period-6 orbit, $a = 0.0125$. (d) Chaos, $a = 0.0158$.

6. Conclusion

We propose dynamical system described by piecewise-defined functions. The Poincaré sections are defined at the break points and the Poincaré mapping is constructed as a composite map of local mappings. We investigated the behavior of the Izhikevich model with state-dependent jump, and computed its bifurcation sets. Bifurcation structure and chaotic parameter regions are clarified in the parameter plane. Moreover, the state of the transition is deeply related to classify characters of the firing patterns in this model.

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