2011 International Symposium on Nonlinear Theory and its Applications
NOLTA2011, Kobe, Japan, September 4-7, 2011

Predator-Prey Dynamics in Hopfield-Type Networks

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Abstract—Predator-prey models have been attracting the interest of researchers in the field of nonlinear dynamics for many decades. In this contribution, we present a novel predator-prey model based on two coupled populations of Hopfield-type neurons. The model exhibits a rich structure of fixed points and periodic and quasi-periodic solutions. We explore it by means of numerical simulations and support our findings with analytical arguments. Furthermore, we show that the equilibrium equations of our model can be understood as meanfield equations of a magnetic spin model. This finding provides an interesting interpretation of predator-prey dynamics in terms of different magnetic phases.

1. Introduction

The study of predator-prey models has a long tradition in the field of non-linear dynamics. Starting from the famous Lotka-Volterra model, a variety of extensions and novel approaches have been introduced [1, 2]. These developments serve as a playground for the study of non-linear phenomena. These models show a considerable diversity of different dynamics, but they all use the same structural elements. They all deal with two interacting populations. The first population suppresses the second population, but the second population encourages the growth of the first population.

In this contribution, we present a model that adopts the basic structure of predator-prey systems and applies it in the context of Hopfield-type neural networks. Each population is modelled by a set of all-to-all coupled Hopfield neurons [4]. The influence of one population on the other population is modelled by a neural meanfield. This allows us to introduce asymmetric couplings in the symmetric Hopfield model. Using symmetry properties, the number of system parameters can be reduced. As a consequence, we obtain a simple system of two coupled difference equations with two system parameters. These equations can be shown to be equivalent to the meanfield equations of an Ising spin model, opening a novel perspective for the study and interpretation of predator-prey models. When studying the system, a varied behaviour of periodic and quasi-periodic orbits is revealed.

In the following sections, we introduce our model and present some first numerical results as well as some supportive analytical results.

2. The Model

We consider a discrete time version of the Hopfield network [4] with constant all-to-all couplings. The dynamics for one population of neurons are governed by the following equations

$$x_i(t+1) = f(\sum_j J x_j(t) + h_i(t)) \tag{1}$$

where $x_i(t) \in [-1, 1]$ is the activity of neuron i at time t, J is the symmetric coupling between neurons, including self-coupling, and h_i is an external input. For the transfer function f we choose $f(.) = \tanh(.)$, which is the smooth version of the sign function that is usually used for Hopfield networks.

The predator-prey situation is modelled by two populations of neurons of sizes N_{pred} and N_{prey}. x^{pred}-type neurons are coupled with J^{pred}, all-to-all within the predator population. x^{prey}-type neurons are coupled with J^{prey} within the prey population.

The neurons of the two populations are not coupled directly. Instead, they are affected by each other's neural meanfields, for which the coupling is positive in the case of predator neurons and negative in the case of prey neurons. Using this trick, we can circumvent the problem of symmetric couplings in Hopfield models and establish the asymmetric interaction which is a property of predator-prey models.

Concretely, each predator neuron is exposed to an additional input of magnitude

$$h^{prey} = J_{c_1} \cdot \overline{x^{prey}}. \tag{2}$$

$\overline{x^{prey}} = \frac{1}{N_{prey}} \sum_{i=1}^{N_{prey}} x_i^{prey}$ is the average acitivity of the prey population and $J_{c_1} > 0$. Similarly, each prey neuron is exposed to

$$h^{pred} = -J_{c_2} \cdot \overline{x^{pred}}. \tag{3}$$

with $J_{c_2} > 0$. The couplings are visualised in Fig.1.

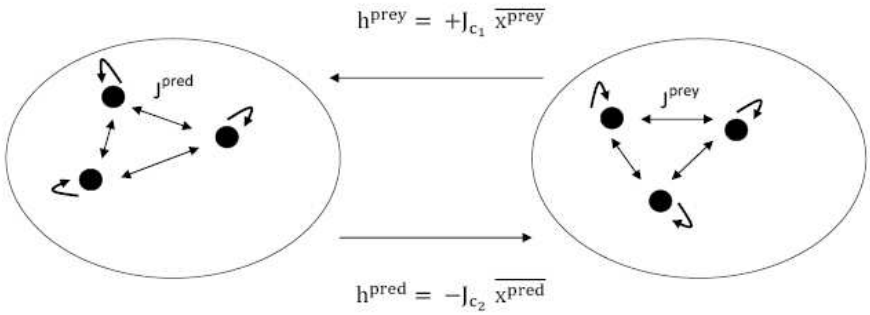

Figure 1: Two Hopfield populations with neural field coupling

In summary, the activity within the network can be described by the following equations

$$\begin{aligned} x_i^{pred}(t+1) &= f\left(\sum_j J^{pred} x_j^{pred}(t) + h^{prey}\right) \\ x_i^{prey}(t+1) &= f\left(\sum_j J^{prey} x_j^{prey}(t) + h^{pred}\right) \end{aligned} \quad (4)$$

For the following examinations we make the assumptions $N = N_{pred} = N_{prey}$, $J_1 = NJ^{pred} = NJ^{prey}$ and $J_2 = J_{c_1} = J_{c_2}$. Then equations (4) can be reduced to a two-dimensional map of the variables $m_1 = \overline{x^{pred}}$ and $m_2 = \overline{x^{prey}}$, if the neurons within each population are initialised uniformly (i.e. $m_1(0) = x_i^{pred}(0)\ \forall i$ and $m_2(0) = x_j^{prey}(0)\ \forall j$). Hence, our final system's equations are

$$\begin{aligned} f_1 : m_1(t+1) &= \tanh\left(J_1 m_1(t) + J_2 m_2(t)\right) \\ f_2 : m_2(t+1) &= \tanh\left(J_1 m_2(t) - J_2 m_1(t)\right) \end{aligned} \quad (5)$$

2.1. Relationship to Spin Systems

Equations (5) are equivalent to a meanfield model of Ising spins. These are binary variables $s_i \in \{-1, 1\}$. In this section, we sketch the main characteristics of this analogy. A detailed discussion will be given elsewhere [6].

In the spin picture, we characterise the predator-prey system as a system of two spin populations $p_1 = \{s_i^{pred}\}$ and $p_2 = \{s_i^{prey}\}$ with a Hamiltoninan (or energy function) H. It is the sum of a predator part H_{pred} and a prey part H_{prey}. Treated in the canonical ensemble, the Hamiltonian determines the probability of a spin configuration $s = (s_1, ..., s_{2N})$ (of both populations) via the Boltzmann distribution

$$p(s) = \frac{1}{Z} e^{-\beta H(s)} \quad (6)$$

where Z is a normalisation constant and the scaling factor can be chosen as $\beta = 1$.

Adopting the coupling convention from above, the Hamiltonian decomposition is

$$\begin{aligned} H_{pred} &= -\sum_{i,j \in p_1} J_1 s_i^{pred} s_j^{pred} - \sum_i J_2 \langle s_i^{prey} \rangle \\ H_{prey} &= -\sum_{i,j \in p_2} J_1 s_i^{prey} s_j^{prey} + \sum_i J_2 \langle s_i^{pred} \rangle \end{aligned} \quad (7)$$

where the sums are over all i, j with $i \neq j$, $\langle s_i \rangle = \sum_{s_i \in \{-1,1\}} p(s_i) s_i$ and $p(s_i)$ is the marginal probability for s_i obtained from $p(s)$.

By applying the meanfield assumption (see [6]), we obtain

$$\begin{aligned} m_1 &= \tanh\left(J_1 m_1(t) + J_2 m_2(t)\right) \\ m_2 &= \tanh\left(J_1 m_2(t) - J_2 m_1(t)\right) \end{aligned} \quad (8)$$

with $m_1 = \overline{\langle s_i^{prey} \rangle}$ and $m_2 = \overline{\langle s_i^{pred} \rangle}$. Hence the meanfield equations (8) correspond to the equilibrium equations of (5). The dynamics described by (5) can be understood as a message passing procedure whose fixed points yield the magnetisations of the spins in the meanfield approximation. This interpretation is not completely new as a relationship betweeen the dynamics in Hopfield networks and the belief propagation message passing algorithm for solutions in the Bethe approximation has already been established [5].

3. Results

3.1. Numeric Simulations

Numeric simulations reveal that the system (5) exhibits stable fixed points as well as stable periodic and quasi-periodic orbits. In Fig. 2, four examples for different parameter settings are shown, using the initial values $m_1 = 0$ and $m_2 = 0.5$.

A more systematic examination shows that

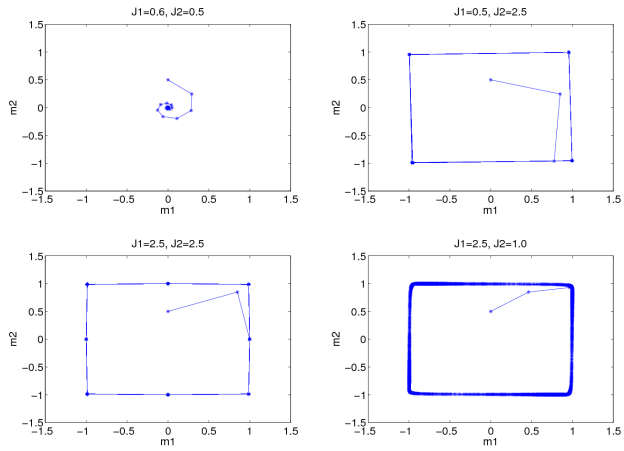

Figure 2: Typical orbits for four different parameter settings: a stable fixed point, stable periodic orbits of periods 4 and 8 and a quasiperiodic orbit.

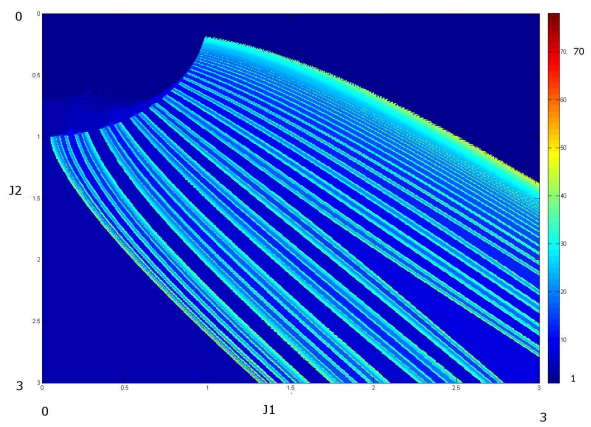

Figure 3: The emergence of stripes of periodic and quasi-periodic solutions in the parameter space. Further explanations are given in the text.

- for $J_1^2 + J_2^2 < 1$ there is one unique stable fixed point $(0, 0)$.
- For large J_1, J_2 and $J_2 >> J_1$ there is a stable period-4 orbit.
- For large J_1, J_2 and $J_1 >> J_2$ there are four stable fixed points near $(1, 1)$,$(-1, -1)$,$(-1, 1)$ and $(1, -1)$.
- There are stripes of periodic and quasiperiodic orbits in between. E.g. for $J_1 = J_2$ (and $J_1 > 1/\sqrt{2}$) we have a stable period-8 orbit.

In Fig. 3, the situation in the parameter space is depicted for the initialisation values $(m_1 = 0, m_2 = 0.5)$. The diagram shows the periodicity in dependence on

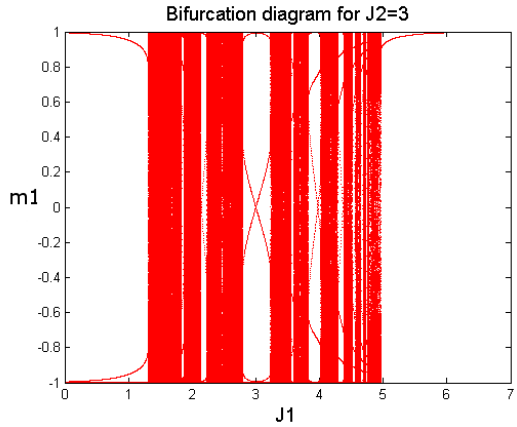

Figure 4: The bifurcation diagram for $J_2 = 3$ in the m_1-projection

the parameter values in colour code. The periodicity was identified via the autocorrelation function, for which the position of the first non-central peak was determined. Although this simple method fails to detect orbits of very high periodicity or quasi-periodic orbits, it yields a qualitatively correct picture of the different periodicity stripes in the parameter space. The existence of areas wtih oribits of extremely high periodicity or with quasi-periodic orbits becomes apparent from phase plots such as the one shown in Fig. 2 or from bifuraction diagrams (Fig. 4).

3.2. Analytical Results

In this section, we derive a couple of analytical results that can explain the larger areas of the plot in Fig. 3.

Proposition 3.1 *For pairs of parameters (J_1, J_2) within the unit circle (i.e. if $J_1^2 + J_2^2 < 1$), the dynamics of equations (5) is a global contraction with fixed point $m_1 = m_2 = 0$.*

Proof: Obviously $m_1 = m_2 = 0$ is a fixed point. We now show that the spectral norm $||Df||$ of the Jacobian Df of (5) is confined by $||Df(0, 0)|| < 1$ if $J_1^2 + J_2^2 < 1$. This is a sufficient condition for a global contraction since $\forall m_1, m_2, \exists \xi : |f(m_1) - f(m_2)| \leq ||Df(\xi)|| \cdot |m_1 - m_2|$ and hence $|f(m_1) - f(m_2)| \leq \lambda |m_1 - m_2|$, $\forall m_1, m_2$ and $\lambda = ||Df(0, 0)|| < 1$. The spectral norm $||A||$ of a real matrix A is defined as the square root of the absolute value of the largest eigenvalue of $A^T A$ [3]. The Jacobian of (5) is

$$Df = \begin{pmatrix} J_1 a & J_2 a \\ -J_2 b & J_2 b \end{pmatrix} \tag{9}$$

with $a = 1 - \tanh^2(J_1 m_1 + J_2 m_2)$ and $b = 1 - \tanh^2(J_1 m_2 - J_2 m_1)$. The matrix $Df^T Df$ is inherently diagonal and the eigenvalues can be read out

directly

$$Df^T Df = \begin{pmatrix} \lambda_1 = (J_1^2 + J_2^2)a^2 & \\ & \lambda_2 = (J_1^2 + J_2^2)b^2 \end{pmatrix} \tag{10}$$

Since $\tanh^2(J_1 m_1 + J_2 m_2) \in [0,1]$, we have $|a| \leq 1$ and $|b| \leq 1$ and $\lambda_{1,2} \leq J_1^2 + J_2^2$ for all (m_1, m_2). The maximum is achieved for $m_1 = m_2 = 0$. Hence $||Df|| \leq ||Df(0,0)|| = \sqrt{J_1^2 + J_2^2} < 1$ and a global contraction is given for $J_1^2 + J_2^2 < 1$ □

Proposition 3.2 *For $J_2 >> J_1 >> 1$ there is a stable period-4 orbit.*

Sketch of the proof: We do not give a full technical proof for this proposition and for the next two propositions. Instead we sketch the proofs in the limit of infintely large couplings and use it as plausibility argument for finite couplings. More elaborated proofs are based on the perturbation theory [6].

Since $J_2 >> J_1$ the maps can be reduced to $f_1 = \tanh(J_2 m_2)$ and $f_2 = -\tanh(J_2 m_1)$. Thus we can decouple the maps when applying twice: $g_1(m_1) = f_1(f_2((m_1)) = -\tanh(J_2 \tanh(J_2 m_1))$ and $g_2(m_2) = f_2(f_1((m_2)) = -\tanh(J_2 \tanh(J_2 m_2))$. Both identical maps have an orbit of period 2 given by $1, -1$ in the limit of very large couplings. They correspond to the following period-4 orbit of the original coupled map: $(1,1), (1,-1), (-1,-1), (-1,1)$. Now we show that $m^* = \pm 1$ are stable fixed points of g_1^2 and g_2^2. For this we need to consider the derivatives, e.g. $(g_{1,2}^2)'(1) = g'_{1,2}(1) \cdot g'_{1,2}(-1)$. Since $|g'_{1,2}(1)| = |g'_{1,2}(-1)| = 0$, the orbit of g is stable and hence the whole period-4 orbit of f is stable.

Proposition 3.3 *For $J_1 >> J_2 >> 1$ there are a four stable fixed points near* $(1,1)$,$(-1,-1)$,$(-1,1)$ *and* $(1,-1)$.

Sketch of the proof: Since $J_1 >> J_2$ the maps can be reduced to the decoupled maps $f_1 = \tanh(J_1 m_1)$ and $f_2 = \tanh(J_1 m_2)$. For each map we find the two fixed points $m_{1,2} = \pm 1$ in the limit of very large couplings. These fixed points are stable since $f'_{1,2}(m_{1,2}) = 0$. Hence the total map f has four stable fixed points which result from the combinations $m_{1,2} = \pm 1$.

Proposition 3.4 *For $J_1 = J_2 >> 1$ there is a stable period-8 orbit.*

Sketch of the proof: It is straightforward to verify that $(1,1)$, $(1,0)$, $(1,-1)$, $(0,-1)$, $(-1,-1)$, $(-1,0)$, $(-1,1)$, $(0,1)$ is a period-8 orbit in the limit of very large couplings. The stability can be verified by considering the Jacobian of f^8. Since it can be decomposed using the chain rule and since all the (partial) derivatives are 0 for the points of the orbit and $J_{1,2} \to \infty$, the orbit must be stable.

4. Discussion

We have presented a novel model of two coupled populations of Hopfield-like neurons, using a predator-prey coupling scheme. Simulations revealed an interesting structure of regions of stable fixed points and stripes of periodic and quasi-periodic solutions. The general structure of the parameter space can be derived from the interpretation of the model in terms of a meanfield spin model:

- For small couplings between the populations and between the individuals ($J_1^2 + J_2^2 < 1$), the system is in the **paramagnetic phase**, where the magnetisation is $m = 0$. The neurons behave as uncoupled individuals.
- For $J_1 >> 0$ and $J_2 \approx 0$, i.e. strong internal couplings and weak couplings between the populations, the system is in the **ferromagnetic phase** ($m = \pm 1$), where each population behaves as an isolated ferromagnet. Hence we observe the four combinations for the magnetisation $(m_1 = 1, m_2 = 1), (1,-1), (-1,1), (-1,-1)$.
- For $J_2, J_1 >> 0$, the system is in a **frustration phase**: the two ferromagnetic populations are frustrating each other due to the asymmetric interaction. For $J_1 = J_2$, the dynamics of the system result in the orbit of period 8, where one state is always flipped: $(1,1)$, $(1,0)$, $(1,-1)$, $(0,-1)$, $(-1,-1)$, $(-1,0)$, $(-1,1)$, $(0,1)$

The analytical examination and characterisation of the periodicity stripes is an interesting issue in our ongoing research.

References

[1] A. A. Berryman, "The origins and evolution of predator-prey theory," *Ecology*, vol.73(5), pp.1530–1535, 1992.

[2] F. Brauer, C. Castillo-Chavez, "Mathematical Models in Population Biology and Epidemiology," *Springer-Verlag, Heidelberg,* 2000.

[3] P. J. S. G. Ferreira, "Fixed point problems - an introduction," Revista do Detuta, Vol.1(6), 1996.

[4] S. Haykin, "Neural networks - a comprehensive introduction," *Prentice-Hall, Inc.*, Second Edition, 1999.

[5] T. Ott, R. Stoop "The neurodynamics of belief propagation on binary Markov random fields," *NIPS 2006*, vol.19, pp.1057–1064, 2007.

[6] In preparation